

# LIFTING REPRESENTATIONS OF FINITE REDUCTIVE GROUPS I: SEMISIMPLE CONJUGACY CLASSES

JEFFREY D. ADLER AND JOSHUA M. LANSKY

**ABSTRACT.** Suppose that  $\tilde{G}$  is a connected reductive group defined over a field  $k$ , and  $\Gamma$  is a finite group acting via  $k$ -automorphisms of  $\tilde{G}$  satisfying a certain quasi-semisimplicity condition (see §3). Then the connected part of the group of  $\Gamma$ -fixed points in  $\tilde{G}$  is reductive. We axiomatize the main features of the relationship between this fixed-point group and the pair  $(\tilde{G}, \Gamma)$ , and consider any group  $G$ , not just the  $\Gamma$ -fixed points of  $\tilde{G}$ , satisfying the axioms. (In fact, the axioms do not require  $\Gamma$  to act on all of  $\tilde{G}$ .) If both  $\tilde{G}$  and  $G$  are  $k$ -quasisplit, then we can consider their duals  $\tilde{G}^*$  and  $G^*$ . We show the existence of and give an explicit formula for a natural map from semisimple stable conjugacy classes in  $G^*(k)$  to those in  $\tilde{G}^*(k)$ . If  $k$  is finite, then our groups are automatically quasisplit, and our result specializes to give a map from semisimple conjugacy classes in  $G^*(k)$  to those in  $\tilde{G}^*(k)$ . Since such classes parametrize packets of irreducible representations of  $G(k)$  and  $\tilde{G}(k)$ , one obtains a mapping of such packets.

## 0. INTRODUCTION

**Motivation.** Suppose that  $F$  is a  $p$ -adic field with residue field  $k$ ;  $E/F$  is a finite, tamely ramified Galois extension;  $H$  is a connected, reductive  $F$ -group; and  $\tilde{H} = R_{E/F}H$  is formed from  $H$  via restriction of scalars. Then one expects to have a *base change lifting* that takes  $L$ -packets of smooth, irreducible representations of  $H(F)$  to  $L$ -packets for  $H(E) = \tilde{H}(F)$ . We would like to gain an explicit understanding, in terms of compact-open data, of base change for depth-zero representations, and this problem requires us to construct a new lifting from (packets of) representations of  $G(k)$  to those of  $\tilde{G}(k)$ , for various connected reductive  $k$ -groups  $G$  and  $\tilde{G}$  attached to parahoric subgroups of  $H(F)$  and  $\tilde{H}(F)$ , respectively. Here  $\Gamma = \text{Gal}(E/F)$  acts on  $\tilde{G}$ , and the connected part of its group of fixed points is  $G$ . (In most cases, this new lifting cannot itself be base change. For more details, see [1, 3].) Since representations of  $G(k)$  can be parametrized by data associated to the dual group  $G^*(k)$ , it is enough to construct an appropriate lifting of such data from  $G^*(k)$  to  $\tilde{G}^*(k)$ .

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*Date:* January 27, 2013.

2010 *Mathematics Subject Classification.* Primary 20G15, 20G40. Secondary 22E50, 20C33, 22E35.

*Key words and phrases.* reductive group, lifting, conjugacy class, representation, Lusztig series.

Both authors were partially supported by the National Science Foundation (DMS-0854844), the National Security Agency (H98230-07-1-002 and H98230-08-1-0068), and Summer Faculty Research Awards from the College of Arts and Sciences of American University.

**This paper.** In the course of creating a candidate for this new lifting, we realized that we could work in greater generality. Namely, let  $k$  denote an arbitrary field,  $\tilde{G}$  and  $G$  connected reductive  $k$ -groups, and  $\Gamma$  a finite group. We assume that  $G$  is a parascopic for  $(\tilde{G}, \Gamma)$  (see Definition 4.1). There are several reasons to make this more general assumption on  $G$ , rather than assuming that  $G = (\tilde{G}^\Gamma)^\circ$ . First, doing so clarifies our proofs. Second, while our original motivation was to improve our explicit understanding of base change for representations of  $p$ -adic groups, our more general formulation appears to be applicable to the understanding of a wider collection of correspondences of representations, including endoscopic transfer. We will take this up elsewhere.

In a very special case of our situation, there is a well-known correspondence between semisimple twisted conjugacy classes in certain groups, and conjugacy classes in subgroups of fixed points (see [4, §6] or [15, §3.10]). However, our map is different and is constructed for different purposes. Indeed, for us,  $G^*$  is usually not a subgroup of  $\tilde{G}^*$ , and we have no notion of twisted conjugacy in the latter. Therefore, neither the results nor the methods *loc. cit.* are relevant in the setting of this paper.

Under our hypotheses, we have the following results.

- (A) Suppose  $\tilde{G}$  and  $G$  are  $k$ -quasisplit. Then we obtain a natural map  $\hat{\mathcal{N}}$  from the  $k$ -variety of semisimple geometric conjugacy classes of the dual  $G^*$  to the analogous variety for  $\tilde{G}^*$  (Proposition 10.1).
- (B) By restricting and refining  $\hat{\mathcal{N}}$ , one obtains a map  $\hat{\mathcal{N}}^{\text{st}}$  from the set of semisimple stable conjugacy classes (in the sense of Kottwitz [10]), in  $G^*(k)$  to that in  $\tilde{G}^*(k)$  (Theorem 11.1).
- (C) If  $k$  is perfect of cohomological dimension  $\leq 1$  (e.g.,  $k$  is finite), then  $\hat{\mathcal{N}}$  is a map from the set of semisimple conjugacy classes in  $G^*(k)$  to that in  $\tilde{G}^*(k)$  (Corollary 11.3).

In order to prove the above results, we require direct knowledge of a special case:  $\Gamma$  acts on  $\tilde{G}$  via  $k$ -automorphisms that all preserve a Borel subgroup of  $\tilde{G}$  and a maximal torus in the Borel subgroup, and  $G$  is the connected part of the group of fixed points of  $\Gamma$  in  $\tilde{G}$ . This “Borel-torus pair” need not be defined over  $k$ . We do require it to be defined over the separable closure  $k^{\text{sep}}$  of  $k$ , but of course this is automatic if  $k$  is perfect. (We show in §12 that this separability hypothesis is quite weak even if  $k$  is imperfect.) Under the above hypotheses, we prove a strong form of the following:

- (D)  $G$  is a reductive  $k$ -group (Proposition 3.5).

We assume that  $\tilde{G}$  and  $G$  are  $k$ -quasisplit in Statements (A) and (B) (it’s automatic under the hypothesis of Statement (C)) for only two reasons. First, we need to know that their  $k$ -duals exist and have good properties. For this purpose, it would be sufficient to have appropriate generalizations of Proposition 6.4 and §8, something that should be achievable for some non-quasisplit groups. Second, we need a way of passing from maximal tori in  $G$  to (stable conjugacy classes of) maximal tori in  $\tilde{G}$ . For this purpose, weaker hypotheses would suffice. For example, it would be sufficient to assume only that  $\tilde{G}$  is  $k$ -quasisplit.

**Outline of this paper.** After establishing some basic notation (§1), we consider how the action of a finite group on a torus  $\tilde{T}$  gives rise to a norm map  $\mathcal{N}: \tilde{T} \rightarrow$

$T$  (where  $T$  is the connected part of the group of fixed points of  $\tilde{T}$ ), and also corresponding maps on the modules of characters and cocharacters of these tori. In fact, we deal with the more general situation where we may replace  $T$  by any torus isogenous to it (§2). We then prove a strong version of Statement (D) above (§3). Suppose  $G$  and  $\tilde{G}$  are connected reductive  $k$ -groups, and  $\Gamma$  is a finite group. In §4, we say what we mean by a *parascopic datum* for the triple  $(\tilde{G}, \Gamma, G)$ , and say that  $G$  is *parascopic* for the pair  $(\tilde{G}, \Gamma)$  if such a datum exists. Given such a datum, we have associated maximal tori  $T \subset G$  and  $\tilde{T} \subset \tilde{G}$ , an action of  $\Gamma$  on the Weyl group  $W(\tilde{G}, \tilde{T})$ , and a canonical embedding  $W(G, T) \rightarrow W(\tilde{G}, \tilde{T})^\Gamma$ , which we describe explicitly in §5. Using standard cohomological arguments, we classify in §6 the set of stable conjugacy classes in a semisimple geometric conjugacy class (of elements), and also the set of stable conjugacy classes of maximal  $k$ -tori. In §7, we define *equivalence* of parascopic data. If  $\tilde{G}$  is quasisplit over  $k$ , then we can prove basic results about equivalence. For example, given a parascopic datum and any maximal  $k$ -torus  $T' \subseteq G$ , there is an equivalent datum associated to  $T'$ . This is crucial in showing that all of our constructions are independent of the choice of a maximal  $k$ -torus in  $G$ . If  $G$  is quasisplit over  $k$ , then we can form its dual group  $G^*$ , and we are then in a position to prove strong duality results (§8) between maximal  $k$ -tori in  $G$  and in  $G^*$ . In particular, up to stable conjugacy, we have a canonical one-to-one correspondence of tori, and this correspondence preserves Weyl groups and much more. If  $\tilde{G}$  is also quasisplit over  $k$ , then we can use this correspondence and our norm map  $\mathcal{N}: \tilde{T} \rightarrow T$  above to define a *conorm map*  $\hat{\mathcal{N}}_{T^*}: T^* \rightarrow \tilde{T}^*$  for dual maximal  $k$ -tori  $T^* \subseteq G^*$  and  $\tilde{T}^* \subseteq \tilde{G}^*$ , and can obtain explicit embeddings of Weyl groups  $W(G^*, T^*) \rightarrow W(\tilde{G}^*, \tilde{T}^*)$  (§9). In particular, we show (Proposition 9.3) that such embeddings have good restriction properties with respect to centralizer subgroups of  $G^*$ . We then have all of the ingredients in place to prove Statement (A) in §10. Using our cohomological results in §6, we can then prove Statement (B) in §11, and it is a simple matter to observe that Statement (C) is just a special case.

In a future work, we will address the problem of lifting other pieces of the parametrization of irreducible representations of finite reductive groups, such as unipotent conjugacy classes in dual groups.

ACKNOWLEDGEMENTS. We have benefited from conversations with Jeffrey Adams, Brian Conrad, Stephen DeBacker, Jeffrey Hakim, Robert Kottwitz, and Jiu-Kang Yu. Thanks also to Bas Edixhoven for pointing out an error in an earlier version of §3 to Brian Conrad for advice on fixing it, and to Avner Ash for coining the term “parascopy”.

## 1. GENERAL NOTATION

Let  $k$  denote a field, and  $k^{\text{sep}}$  a separable closure of  $k$ . We will abbreviate  $\text{Gal}(k^{\text{sep}}/k)$  by  $\text{Gal}(k)$ . Given a connected reductive  $k$ -group  $G$  and a maximal torus  $T$  of  $G$ , let  $\Phi(G, T)$  (resp.  $\Phi^\vee(G, T)$ ) denote the absolute root (resp. coroot) system of  $G$  with respect to  $T$ . Given  $\alpha \in \Phi(G, T)$ , let  $\alpha^\vee$  denote the corresponding coroot in  $\Phi^\vee(G, T)$ . Let  $W(G, T)$  denote the Weyl group of  $G$  with respect to  $T$ .

For  $g \in G(k^{\text{sep}})$ , let  $\text{Int}$  denote the natural homomorphism  $G \rightarrow \text{Inn}(G)$  given by  $\text{Int}(g)(x) = gxg^{-1}$ . If  $x \in G(k^{\text{sep}})$  (resp.  $X \subseteq G$ ), we will also denote  $\text{Int}(g)(x)$  (resp.  $\text{Int}(g)(X)$ ) by  ${}^gx$  (resp.  ${}^gX$ ).

If  $\phi$  is a homomorphism from a group  $\Gamma$  to the group of automorphisms of some object, then we will denote the operation of taking  $\phi(\Gamma)$ -fixed points by  $(\ )^{\phi(\Gamma)}$ , or just by  $(\ )^\Gamma$  when  $\phi$  is understood.

For any  $k^{\text{sep}}$ -torus  $T$ , let  $\mathbf{X}^*(T)$  and  $\mathbf{X}_*(T)$  respectively denote the character and cocharacter modules of  $T$ . Let  $\langle \ , \ \rangle$  denote the natural bilinear pairing between  $\mathbf{X}^*(T)$  and  $\mathbf{X}_*(T)$ . Let  $V^*(T) = \mathbf{X}^*(T) \otimes \mathbb{Q}$  and  $V_*(T) = \mathbf{X}_*(T) \otimes \mathbb{Q}$ . Then  $\langle \ , \ \rangle$  extends to a nondegenerate pairing between the  $\mathbb{Q}$ -vector spaces  $V^*(T)$  and  $V_*(T)$ . Any homomorphism  $f: T \rightarrow T'$  of tori determines maps  $f^*: \mathbf{X}^*(T') \rightarrow \mathbf{X}^*(T)$  and  $f_*: \mathbf{X}_*(T) \rightarrow \mathbf{X}_*(T')$ , and hence maps  $V^*(T') \rightarrow V^*(T)$  and  $V_*(T) \rightarrow V_*(T')$  that we will also denote by  $f^*$  and  $f_*$ , respectively.

For an algebraic  $k$ -group  $G$ , let  $G^\circ$  denote its connected part. For  $i = 1, 2$ , let  $T_i$  be a maximal torus of  $G$ , and suppose that  ${}^g T_1 = T_2$  for some  $g \in G(k^{\text{sep}})$ . Then  $\text{Int}(g)$  gives an isomorphism  $T_1 \rightarrow T_2$ . For  $\chi \in \mathbf{X}^*(T_1)$  and  $\lambda \in \mathbf{X}_*(T_1)$ , define

$${}^g \chi := \text{Int}(g)^{* -1} \chi \quad {}^g \lambda := \text{Int}(g)_* \lambda.$$

## 2. FINITE-GROUP ACTIONS ON CHARACTER AND COCHARACTER MODULES

Let  $\tilde{T}$  be a  $k$ -torus and let  $\Gamma$  be a finite group that acts on  $\tilde{T}$  via  $k$ -automorphisms. Then  $\Gamma$  acts on both  $\mathbf{X}^*(\tilde{T})$  and  $\mathbf{X}_*(\tilde{T})$  (and thus on  $V^*(\tilde{T})$  and  $V_*(\tilde{T})$ ) via the rules

$$\gamma \cdot \chi = \gamma^{*-1} \chi, \quad \gamma \cdot \lambda = \gamma_* \lambda$$

for  $\chi \in \mathbf{X}^*(\tilde{T})$  and  $\lambda \in \mathbf{X}_*(\tilde{T})$ .

Suppose that  $T$  is another  $k$ -torus, and  $j_*: V_*(T) \rightarrow V_*(\tilde{T})^\Gamma$  is an isomorphism. Composing  $j_*$  with the natural inclusion  $V_*(\tilde{T})^\Gamma \rightarrow V_*(\tilde{T})$ , and taking duals, we obtain maps  $i_*$ ,  $j^*$ , and  $i^*$  as follows:

$$\begin{array}{ccc} & \xrightarrow{i_*} & \\ V_*(T) & \xrightarrow[\sim]{j_*} V_*(\tilde{T})^\Gamma \hookrightarrow V_*(\tilde{T}) & \\ & \xleftarrow[\pi]{\sim} & \end{array} \quad \begin{array}{ccc} & \xleftarrow{i^*} & \\ V^*(T) & \xleftarrow[\sim]{j^*} V^*(\tilde{T})_\Gamma \leftarrow V^*(\tilde{T}) & \\ & \xleftarrow[\iota]{\sim} & \end{array}$$

where  $\iota$  and  $\pi$  are to be described below.

Define a map  $\pi: V_*(\tilde{T}) \rightarrow V_*(T)$  by

$$\pi(w) := j_*^{-1} \left( \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot w \right).$$

We have that  $\pi \circ i_* = \text{id}$ , so  $\pi$  is a projection of  $V_*(\tilde{T})$  onto  $V_*(T)$ . The kernel of  $\pi$  is the span of the set of vectors in  $V_*(\tilde{T})$  of the form  $w - \gamma_* w$ , where  $w \in V_*(\tilde{T})$  and  $\gamma \in \Gamma$ .

Let  $\iota: V^*(T) \rightarrow V^*(\tilde{T})$  be the transpose of  $\pi$ . That is,  $\iota$  is the map that satisfies  $\langle \iota(v), w \rangle = \langle v, \pi(w) \rangle$  for  $v \in V^*(T)$ ,  $w \in V_*(\tilde{T})$ . More explicitly, if  $\tilde{v} \in V^*(\tilde{T})$  is any preimage under  $i^*$  of  $v$ , then

$$\begin{aligned} \langle \iota(v), w \rangle &= \langle v, \pi(w) \rangle = \langle i^* \tilde{v}, \pi(w) \rangle = \langle \tilde{v}, i_* \pi(w) \rangle \\ &= \left\langle \tilde{v}, \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot w \right\rangle = \left\langle \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot \tilde{v}, w \right\rangle. \end{aligned}$$

Thus

$$(2.1) \quad \iota(v) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot \tilde{v}.$$

The image of  $\iota$  is clearly the subspace  $V^*(\tilde{T})^\Gamma$  of  $\Gamma$ -fixed vectors in  $V^*(\tilde{T})$ . Moreover,  $\iota$  is injective since  $\pi$  is surjective, and it respects the bilinear pairings associated to both  $T$  and  $\tilde{T}$  in the sense that for all  $v \in V^*(T)$  and  $w \in V_*(T)$ , we have  $\langle \iota(v), i_*(w) \rangle = \langle v, w \rangle$ . Using  $\iota$  (resp.  $i_*$ ), we may therefore identify  $V^*(T)$  (resp.  $V_*(T)$ ) with  $V^*(\tilde{T})^\Gamma$  (resp.  $V_*(\tilde{T})^\Gamma$ ). We note that

$$(2.2) \quad i^* = \iota^{-1} \circ \left( \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \right).$$

Define a map  $\mathcal{N}_*: V_*(\tilde{T}) \rightarrow V_*(T)$  by

$$(2.3) \quad \mathcal{N}_* = j_*^{-1} \circ \left( \sum_{\gamma \in \Gamma} \gamma_* \right) = |\Gamma| \pi.$$

Taking duals, we have a map  $\mathcal{N}^*: V^*(T) \rightarrow V^*(\tilde{T})$  given by

$$\mathcal{N}^*v = \sum_{\gamma \in \Gamma} \gamma \cdot \tilde{v},$$

where  $\tilde{v}$  is any preimage under  $i^*$  of  $v$ .

Note that if  $j_*$  is equivariant under  $\text{Gal}(k)$ , then so are all of the other maps above.

*Example 2.4.* Let  $T = (\tilde{T}^\Gamma)^\circ$ . Then we have an inclusion map  $i: T \rightarrow \tilde{T}$ , and a norm map  $\mathcal{N}_T = \mathcal{N}: \tilde{T} \rightarrow T$  given by

$$(2.5) \quad \mathcal{N}(t) = \prod_{\gamma \in \Gamma} \gamma(t).$$

The maps  $i_*$  and  $i^*$  are induced by  $i$ , and the maps  $\mathcal{N}_*$  and  $\mathcal{N}^*$  are induced by  $\mathcal{N}$ .

### 3. FINITE-GROUP ACTIONS ON REDUCTIVE GROUPS

In this section, we establish a stronger form of Statement (D) from the Introduction.

**Definition 3.1.** We say that a  $k$ -automorphism  $\gamma$  of a connected reductive  $k$ -group  $\tilde{G}$  is *quasi-semisimple* if  $\gamma$  preserves a Borel subgroup  $\tilde{B}_\bullet$  of  $\tilde{G}$  and a maximal torus  $\tilde{T}_\bullet$  in  $\tilde{B}_\bullet$ . We say that  $\gamma$  is *separably quasi-semisimple* if the groups  $\tilde{B}_\bullet$  and  $\tilde{T}_\bullet$  can be chosen to be defined over  $k^{\text{sep}}$ .

*Remark 3.2.* Suppose  $\gamma$  is a quasi-semisimple  $k$ -automorphism of  $\tilde{G}$ . If  $k$  is an imperfect field of characteristic two, and  $\tilde{G}$  has a factor of type  $A_{2n}$  on which some power of  $\gamma$  acts via a non-inner automorphism, then we do not know if  $\gamma$  must be separably quasi-semisimple. However, we will show in §12 that in all other cases, it must.

**Lemma 3.3.** *Suppose  $\tilde{G}$  is a connected reductive  $k$ -group, and  $\gamma$  is a quasi-semisimple  $k$ -automorphism of  $\tilde{G}$ . Let  $\tilde{B}_\bullet$  be a  $\gamma$ -invariant Borel subgroup of  $\tilde{G}$ , and  $\tilde{T}_\bullet$  a  $\gamma$ -invariant maximal torus of  $\tilde{G}$ . Then the group  $G := (\tilde{G}^\gamma)^\circ$  is reductive,*

$T_\bullet := (\tilde{T}_\bullet^\gamma)^\circ = G \cap \tilde{T}_\bullet$  is a maximal torus in  $G$ , and  $B_\bullet := \tilde{B}_\bullet^\gamma$  is a Borel subgroup of  $G$  containing  $T_\bullet$ .

*Proof.* There exist a simply connected group  $\tilde{G}'$ , a torus  $\tilde{Z}$ , and a central isogeny  $\phi: \tilde{Z} \times \tilde{G}' \rightarrow \tilde{G}$ . Then  $\gamma$  lifts uniquely to an automorphism of  $\tilde{Z} \times \tilde{G}'$ . Moreover,  $\tilde{G}'$  contains a maximal torus  $\tilde{T}'_\bullet$  and Borel subgroup  $\tilde{B}'_\bullet$  such that  $\tilde{Z} \times \tilde{T}'_\bullet$  and  $\tilde{Z} \times \tilde{B}'_\bullet$  are the inverse images under  $\phi$  of  $\tilde{T}_\bullet$  and  $\tilde{B}_\bullet$ , and  $\gamma$  preserves these groups. From Steinberg [18, Theorem 8.2], we know that  $G' := \tilde{G}'^\gamma$  is a reductive group, and it is clear from [*loc. cit.*, Remark 8.3(a)] that  $T'_\bullet := (\tilde{T}'_\bullet)^\gamma = G' \cap \tilde{T}'_\bullet$  is a maximal torus in  $G'$  and  $B'_\bullet := (\tilde{B}'_\bullet)^\gamma$  is a Borel subgroup of  $G'$  containing  $T'_\bullet$ . We obtain our desired result by taking images under  $\phi$  of these groups.  $\square$

**Lemma 3.4.** *Suppose  $\tilde{G}$  is a connected reductive  $k$ -group, and  $\gamma$  is a separably quasi-semisimple  $k$ -automorphism of  $G$ . Then the group  $G := (\tilde{G}^\gamma)^\circ$  is defined over  $k$ .*

*Proof.* Let  $\tilde{T}_\bullet$  and  $\tilde{B}_\bullet$  be as in Definition 3.1, and we assume that these groups are defined over  $k^{\text{sep}}$ . Since  $\gamma$  is defined over  $k$ , we have that  $\text{Gal}(k)$  preserves  $G$ . Therefore, it will be enough to show that  $G$  is defined over  $k^{\text{sep}}$ . We may therefore assume that  $k = k^{\text{sep}}$  is separably closed.

Let  $T_\bullet$  and  $B_\bullet$  be as in Lemma 3.3. Consider the free part of module of  $\gamma$ -coinvariants in the lattice  $\mathbf{X}^*(\tilde{T}_\bullet)$ . Since this is a lattice, and it is a quotient of  $\mathbf{X}^*(\tilde{T}_\bullet)$ , it is the character lattice of a  $k$ -subtorus of  $\tilde{T}_\bullet$ , and this subtorus is precisely  $T_\bullet$ . That is,  $T_\bullet$  is defined over  $k$ . Since  $\tilde{T}_\bullet$  is split over  $k$ , the root groups  $U_\alpha$  are defined over  $k$  for each  $\alpha \in \Phi(\tilde{G}, \tilde{T}_\bullet)$ , and there exist  $k$ -isomorphisms  $x_\alpha$  from the additive group to each  $U_\alpha$ . Moreover, if  $\alpha$  is a root whose  $\gamma$ -orbit has size  $r$ , then since  $\gamma$  is defined over  $k$ , we may select the automorphisms  $x_{\gamma^i \alpha}$  so that  $\gamma^i \circ x_\alpha = x_{\gamma^i \alpha}$  for  $i = 0, \dots, r-1$ . It follows from the description of the root groups for  $T_\bullet$  given, say, in part (2''') of the proof of [18, Theorem 8.2], that each such root group  $U_\beta$  ( $\beta \in \Phi(G, T_\bullet)$ ) inherits a  $k$ -structure from that of  $\tilde{G}$ . Thus, the product  $Y = T_\bullet \prod U_\beta$  of  $T_\bullet$  with all of the root groups  $U_\beta$  (in any order) is an open  $k$ -variety in  $G$ . As a result,  $Y(k) \subset G \cap \tilde{G}(k)$  is dense in  $Y$ , and hence in  $G$ . It follows from [17, Lemma 11.2.4] that  $G$  is defined over  $k$ .  $\square$

**Proposition 3.5.** *Suppose  $\tilde{G}$  is a connected reductive  $k$ -group, and  $\Gamma$  is a finite group that acts on  $\tilde{G}$  via  $k$ -automorphisms that preserve a Borel  $k^{\text{sep}}$ -subgroup  $\tilde{B}_\bullet$  of  $\tilde{G}$  and a maximal  $k^{\text{sep}}$ -torus  $\tilde{T}_\bullet$  in  $\tilde{B}_\bullet$ . Let  $G = (\tilde{G}^\Gamma)^\circ$ . Then:*

- (a)  $G$  is a reductive  $k$ -group.
- (b) For every Borel-torus pair  $(\tilde{B}, \tilde{T})$  in  $\tilde{G}$  preserved by  $\Gamma$ , we have that  $(\tilde{B}^\Gamma, (\tilde{T}^\Gamma)^\circ)$  is a Borel-torus pair for  $G$ .
- (c) Let  $T$  be a maximal torus in  $G$ , and let  $\tilde{T} = C_{\tilde{G}}(T)$ . Then  $\tilde{T}$  is a maximal torus in  $\tilde{G}$ .
- (d) Let  $T, \tilde{T}$  be as in (c). Then each root in  $\Phi(G, T)$  is the restriction to  $T$  of a root in  $\Phi(\tilde{G}, \tilde{T})$ .
- (e) Let  $\tilde{T}$  be as in (c). Then there is some Borel subgroup  $\tilde{B}$  of  $\tilde{G}$  containing  $\tilde{T}$  such that  $(\tilde{B}, \tilde{T})$  is a Borel-torus pair preserved by  $\Gamma$ .

*Remarks 3.6.*

- (i) Part (a) is proved by Prasad and Yu [12] under somewhat different hypotheses. Rather than assuming that  $\Gamma$  fixes a Borel-torus pair, they assume that  $|\Gamma|$  is not divisible by  $\text{char } k$ . When  $\text{char } k = 0$ , their hypotheses are strictly weaker than ours.
- (ii) One can choose  $T$  in part (c) to be defined over  $k$ , in which case the torus  $\tilde{T}$  is also defined over  $k$ .

*Remark 3.7.* Note that this proposition holds for  $\tilde{G}$  if and only if it holds for  $\tilde{G}/Z(\tilde{G})$ . Moreover, if  $\tilde{G}$  is semisimple, then the action of  $\Gamma$  lifts to an action on the simply connected cover of  $\tilde{G}$ , so the previous sentence implies that we may assume that  $\tilde{G}$  is simply connected when convenient to do so.

*Proof.* Let  $(\tilde{B}_\bullet, \tilde{T}_\bullet)$  denote a  $\Gamma$ -invariant Borel-torus pair in  $\tilde{G}$ . By Remark 3.7, we may assume that  $\tilde{G}$  has trivial center.

We first prove (c) and (e). Let  $T_\bullet$  be the torus  $(\tilde{T}_\bullet^\Gamma)^\circ$ . We first seek to show that  $T_\bullet$  is maximal in  $G$  and  $C_{\tilde{G}}(T_\bullet) = \tilde{T}_\bullet$ . We prove the latter by showing that the root system  $\Phi(C_{\tilde{G}}(T_\bullet), \tilde{T}_\bullet)$  is empty. Let  $\Phi^+$  denote the positive subsystem of  $\Phi(C_{\tilde{G}}(T_\bullet), \tilde{T}_\bullet)$  determined by  $\tilde{B}_\bullet$ . Suppose for a contradiction that  $\Phi^+$  is nonempty, and choose  $\alpha \in \Phi^+$ . Let  $\chi = \sum_{\gamma \in \Gamma} \gamma \cdot \alpha$ . Since  $\Gamma$  preserves  $\tilde{B}_\bullet$ , we have that  $\Gamma$  preserves  $\Phi^+$ , so  $\chi$  is a positive linear combination of elements of  $\Phi^+$ , and is thus nonzero.

The canonical pairing  $\langle \cdot, \cdot \rangle$  between  $\mathbf{X}_*(\tilde{T}_\bullet)$  and  $\mathbf{X}^*(\tilde{T}_\bullet)$  is invariant under the action of  $\Gamma$ . Thus, for all  $\lambda \in \mathbf{X}_*(T_\bullet) = \mathbf{X}_*(\tilde{T}_\bullet)^\Gamma$ ,  $\langle \chi, \lambda \rangle = \sum_{\gamma \in \Gamma} \langle \gamma \cdot \alpha, \lambda \rangle = |\Gamma| \langle \alpha, \lambda \rangle$  which is 0 since  $\alpha$  is a root of the centralizer of  $T_\bullet$ . Since  $\chi$  is clearly  $\Gamma$ -invariant, we may identify it with a vector  $\chi_0 \in V^*(T_\bullet)$  via the map  $\iota^{-1}$  as discussed in §2. (In fact,  $\chi_0 = |\Gamma| i^* \alpha$  by (2.2).) Then  $\langle \chi_0, \lambda \rangle = \langle \chi, \lambda \rangle$  for all  $\lambda \in \mathbf{X}_*(T_\bullet)$ . Since the pairing between  $\mathbf{X}^*(T_\bullet)$  and  $\mathbf{X}_*(T_\bullet)$  is nondegenerate, it follows that  $\chi = 0$ , a contradiction.

To see that  $T_\bullet$  is maximal in  $G$ , consider a maximal torus  $T'_\bullet$  containing  $T_\bullet$ . Then

$$T'_\bullet \subseteq C_{\tilde{G}}(T'_\bullet) \subseteq C_{\tilde{G}}(T_\bullet) = \tilde{T}_\bullet.$$

Taking connected parts of groups of  $\Gamma$ -fixed points, we see that  $T'_\bullet \subseteq T_\bullet$ , and thus the two tori are equal.

Now let  $T$  denote an arbitrary maximal torus in  $G$ . We can write  $T = {}^g T_\bullet$  for some  $g \in G(\bar{k})$ . Thus,  $C_{\tilde{G}}(T) = {}^g \tilde{T}_\bullet$ , which is a maximal torus in  $\tilde{G}$ . Also,  ${}^g \tilde{B}_\bullet$  is a  $\Gamma$ -invariant Borel subgroup of  $\tilde{G}$  containing  $\tilde{T}$ . This proves (c) and (e).

We now prove the other statements of the theorem simultaneously by a series of reductions. By Remark 3.7, we may assume that  $\tilde{G}$  is simply connected.

Suppose  $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2$ , a direct product of connected reductive  $k$ -groups, each preserved by  $\Gamma$ . By induction on the dimension of  $\tilde{G}$ , we see that each statement holds for each  $\tilde{G}_i$ , and thus for  $\tilde{G}$ . It thus only remains to consider the case where  $\tilde{G}$  has no such decomposition. That is, we may assume that  $\tilde{G} = \prod_{i=1}^r \tilde{G}_i$ , a direct product of  $k$ -simple factors which are permuted transitively by  $\Gamma$ .

For  $1 < i \leq r$ , choose  $\gamma_i \in \Gamma$  such that  $\gamma_i(\tilde{G}_1) = \tilde{G}_i$ . Let  $\gamma_1$  equal the identity in  $\Gamma$ . Use each  $\gamma_i$  to identify  $\tilde{G}_1$  with  $\tilde{G}_i$ . Thus, we have a diagonal embedding

$$\text{diag}: \tilde{G}_1 \longrightarrow \prod_i \tilde{G}_1 = \tilde{G},$$

and  $\tilde{G}^\Gamma \subseteq \text{diag}(\tilde{G}_1) = \tilde{G}^{S_r}$ , where the symmetric group  $S_r$  acts here by permuting the coordinates of elements of  $\tilde{G}$ . For  $\gamma \in \Gamma$  and  $g \in \tilde{G}_1(\bar{k})$ , we have

$$\gamma(\text{diag}(g)) = (\theta_{\gamma,1}(g), \dots, \theta_{\gamma,r}(g)),$$

where each  $\theta_{\gamma,i}$  belongs to  $\text{Aut}_k(\tilde{G}_1)$ . Note that  $\theta_{\gamma,i} = \theta_{\gamma_i^{-1}\gamma,1}$ , so the set  $\Gamma_i := \{\theta_{\gamma,i} \mid \gamma \in \Gamma\}$  is independent of  $i$ . Moreover, we can see that  $\Gamma_1$  is the image (under restriction to  $\tilde{G}_1$ ) of  $\text{stab}_\Gamma(\tilde{G}_1)$ , and is thus a group.

Therefore,

$$\tilde{G}^\Gamma = \text{diag}(\tilde{G}_1^{\Gamma_1}) = \text{diag}(\tilde{G}_1)^{\Gamma_1} = \tilde{G}^{S_r \times \Gamma_1},$$

where  $\Gamma_1$  acts on  $\tilde{G} = \prod_i \tilde{G}_1$  in the same way on each factor. Thus, we may replace  $\Gamma$  by  $S_r \times \Gamma_1$ .

Using induction on  $|\Gamma|$  and working in stages, we see that we may always replace  $\Gamma$  with the successive subquotients that occur in any subnormal series for  $\Gamma$ . If  $\Gamma$  is just a permutation action on the factors of  $\tilde{G}$ , then the proposition is clear. Thus, we may assume that  $\tilde{G}$  is  $k$ -simple and  $\Gamma$  is simple.

Then we may write  $\tilde{G} = \prod_{j=1}^m \tilde{M}_j$  where each  $\tilde{M}_j$  is an absolutely simple  $k^{\text{sep}}$ -group, and  $\text{Gal}(k)$  acts on the set of factors via a transitive subgroup of the symmetric group  $S_m$ . Every element of  $\Gamma$  induces a permutation of the factors, thus giving a map from  $\Gamma$  to the symmetric group  $S_m$ . Since  $\Gamma$  is simple, this map is either trivial or injective.

Suppose that the map  $\Gamma \rightarrow S_m$  is injective. Then since  $\Gamma$  commutes with the (transitive) image of  $\text{Gal}(k)$  in  $S_m$ , we have that for every  $1 \leq i < j \leq m$  in the same  $\Gamma$ -orbit, there is only one element of  $\Gamma$  that takes  $\tilde{M}_i$  to  $\tilde{M}_j$ . It is then easy to see that  $G$  is a product of groups isomorphic to some (hence any)  $\tilde{M}_j$ , and thus  $G$  is reductive. Since  $G$  is defined over  $k^{\text{sep}}$  and preserved by  $\text{Gal}(k)$ , it is defined over  $k$ .

On the other hand, suppose that the map  $\Gamma \rightarrow S_m$  is trivial. Thus,  $\Gamma$  preserves each factor  $\tilde{M}_j$ . Since  $\Gamma$  is simple, and since  $\Gamma$  commutes with the (transitive) image of  $\text{Gal}(k)$  in  $S_m$ , it either consists of inner automorphisms or it is isomorphic to a subgroup of  $\prod \Gamma_j$ , where each  $\Gamma_j$  now denotes the symmetry group of the absolute Dynkin diagram of  $\tilde{M}_j$ . It will be enough to show that  $\Gamma$  is solvable, since then it must be cyclic. In this case, (a) and (b) follow from Lemma 3.3 and Lemma 3.4, and (d) follows from Remark 3.7 and the description of the root groups of  $G$  in [18, §8.2(2''')].

If  $\Gamma$  consists of inner automorphisms, then  $\Gamma$  is a subgroup of  $\tilde{B}_\bullet$ , and is thus solvable. On the other hand, suppose  $\Gamma \subseteq \prod \Gamma_j$ . Recall that all connected Dynkin diagrams have solvable symmetry groups. Thus, once again,  $\Gamma$  is solvable.  $\square$

**Corollary 3.8.** *Let  $\tilde{T} \subseteq \tilde{G}$  be any  $\Gamma$ -invariant maximal torus contained in a  $\Gamma$ -invariant Borel subgroup of  $\tilde{G}$ . Then the natural map from  $W(\tilde{G}, \tilde{T})^\Gamma$  to the group  $\text{Aut}_{\mathbb{Q}}(V^*(\tilde{T})^\Gamma)$  is injective.*

*Proof.* Let  $G = (\tilde{G}^\Gamma)^\circ$  and let  $T = (\tilde{T}^\Gamma)^\circ$ . Suppose  $n \in \tilde{G}$  is a representative for an element  $w \in W(\tilde{G}, \tilde{T})^\Gamma$ . Then  $n$  acts on  $\tilde{T}^\Gamma$ , and thus normalizes  $T$ . Applying the results of §2 according to Example 2.4, we may identify  $V^*(\tilde{T})^\Gamma$  and  $V^*(T)$  via the isomorphism  $\iota$ , as described in (2.1). Observe that  $\iota$  commutes with the action of  $n$ . Suppose further that  $n$  acts trivially on  $V^*(\tilde{T})^\Gamma$ , and thus on  $V^*(T)$ . Then



$n$  acts trivially on  $T$ , so  $n \in C_{\tilde{G}}(T)$ , which by Proposition 3.5(b,c) is equal to  $\tilde{T}$ . Thus,  $n$  acts trivially on  $\tilde{T}$ .  $\square$

#### 4. PARASCOPY: DEFINITION

We now axiomatize the essential properties of the relationship between the group  $G$  and the pair  $(\tilde{G}, \Gamma)$  of §3.

**Definition 4.1.** Let  $G$  and  $\tilde{G}$  be connected reductive  $k$ -groups, and  $\Gamma$  a finite group. A *parascopic datum* for the triple  $(\tilde{G}, \Gamma, G)$  is a pair  $(\phi, j_*)$ , where

- $\phi$  is a homomorphism from  $\Gamma$  to  $\text{Aut}_k(\tilde{T})$  for some maximal  $k$ -torus  $\tilde{T} \subseteq \tilde{G}$ , such that some positive system of roots in  $\Phi(\tilde{G}, \tilde{T})$  is preserved by all automorphisms in  $\phi(\Gamma)$ ; and
- $j_*: V_*(T) \rightarrow V_*(\tilde{T})^{\phi(\Gamma)}$  is a  $\text{Gal}(k)$ -equivariant isomorphism for some maximal  $k$ -torus  $T \subseteq G$ ;

satisfying two conditions.

$$\mathbf{P1.} \quad j_*(\mathbf{X}_*(T)) \supseteq \mathbf{X}_*(\tilde{T})^{\phi(\Gamma)}.$$

Composing  $j_*$  with the inclusion map  $V_*(\tilde{T})^{\phi(\Gamma)} \rightarrow V_*(\tilde{T})$ , we obtain a map  $i_*: V_*(T) \rightarrow V_*(\tilde{T})$  whose dual  $i^*: V^*(\tilde{T}) \rightarrow V^*(T)$  is assumed to satisfy:

$$\mathbf{P2.} \quad i^*(\Phi(\tilde{G}, \tilde{T})) \supseteq \Phi(G, T).$$

We will say that  $G$  is a *parascopic group* for the pair  $(\tilde{G}, \Gamma)$  if a parascopic datum for  $(\tilde{G}, \Gamma, G)$  exists. We will say that the tori  $T$  and  $\tilde{T}$  appearing in this definition are the *implicit tori* for the datum in  $G$  and  $\tilde{G}$ , respectively.

*Examples 4.2.*

- Suppose  $\Gamma$  acts separably quasi-semisimply on  $\tilde{G}$ , and  $G = (\tilde{G}^\Gamma)^\circ$ , as in §3. From Proposition 3.5 and Remark 3.6(ii), we can choose maximal  $k$ -tori  $T \subseteq G$  and  $\tilde{T} \subseteq \tilde{G}$  such that  $T = (\tilde{T}^\Gamma)^\circ$ , and the given action  $\phi: \Gamma \rightarrow \text{Aut}_k(\tilde{T})$ , together with the map  $j_*$  induced by the inclusion  $T \rightarrow \tilde{T}$ , form a parascopic datum.
- If  $G$  is a Levi subgroup of  $\tilde{G}$ , then  $G$  is clearly parascopic for  $(\tilde{G}, 1)$ .
- If  $G$  is the image under a separable  $k$ -isogeny of a parascopic group for  $(\tilde{G}, \Gamma)$ , then  $G$  is itself parascopic for  $(\tilde{G}, \Gamma)$ .
- Our definition does not refer to any action of  $\Gamma$  on  $\tilde{G}$ , but it turns out that one can indeed lift  $\phi$  to a map  $\Gamma \rightarrow \text{Aut}_k(\tilde{G})$ . While different choices of lifting can lead to groups of fixed points  $\tilde{G}^{\phi(\Gamma)}$  whose connected parts are non-isomorphic, these differences do not matter for our present purposes, and thus there is no need to make such choices. We will show elsewhere [2] that a particular lifting  $\phi_0: \Gamma \rightarrow \text{Aut}_k(\tilde{G})$  has the following property: If  $\phi$  denotes any lifting, then  $(G^{\phi(\Gamma)})^\circ$  is parascopic for  $((G^{\phi_0(\Gamma)})^\circ, 1)$  in a natural way. For example,  $\text{SO}(2n)$  is parascopic for  $(\text{Sp}(2n), 1)$ .

In §7, we will define and study a natural notion of equivalence between parascopic data.

## 5. FINITE-GROUP ACTIONS AND WEYL GROUPS

Suppose that  $\tilde{G}$  and  $G$  are connected reductive  $k$ -groups and  $\Gamma$  is a finite group. Let  $(\phi, j^*)$  be a parascopic datum for  $(\tilde{G}, \Gamma, G)$ . As indicated in §1, to ease notation, we will suppress reference to  $\phi$  when considering the action of  $\Gamma$ . From now on, use the maps  $i_*$  and  $\iota$  of §2 to identify  $V_*(T)$  with  $V_*(\tilde{T})^\Gamma$  and  $V^*(T)$  with  $V^*(\tilde{T})^\Gamma$ .

In the particular situation where  $G = (\tilde{G}^\Gamma)^\circ$ , there is an obvious embedding  $W(G, T) \longrightarrow W(\tilde{G}, \tilde{T})^\Gamma$ . Here  $w \in W(G, T)$  corresponds to the unique element of  $W(\tilde{G}, \tilde{T})^\Gamma$  whose action when restricted to  $T$  coincides with that of  $w$ . We now show that we have such an embedding in our more general situation, where  $G$  is only assumed to be parascopic.

Let  $\alpha$  be a root in  $\Phi(G, T)$ . Then  $\alpha = i^*\tilde{\alpha}$  for some root  $\tilde{\alpha}$  in  $\Phi(\tilde{G}, \tilde{T})$  by Condition P2 in the definition of parascopic datum. There are two cases to consider:

- (1) The roots in  $\Gamma \cdot \tilde{\alpha}$  are mutually orthogonal.
- (2) The roots in  $\Gamma \cdot \tilde{\alpha}$  are not mutually orthogonal.

In case (1), let  $\Psi = \Gamma \cdot \tilde{\alpha}$ .

In case (2), [11, §1.3] implies that for each  $\theta \in \Gamma \cdot \tilde{\alpha}$ , there exists a unique root  $\theta' \neq \theta$  in  $\Gamma \cdot \tilde{\alpha}$  such that  $\theta$  and  $\theta'$  are not orthogonal. Moreover,  $\theta + \theta'$  is a root in  $\Phi(\tilde{G}, \tilde{T})$ . Let  $\Psi = \{\theta + \theta' \mid \theta \in \Gamma \cdot \tilde{\alpha}\}$ .

*Remark 5.1.* Although it is assumed in [11] that  $\Gamma$  is cyclic, there is only one case in which the action of the stabilizer in a general group  $\Gamma$  of an irreducible component of  $\Phi(\tilde{G}, \tilde{T})$  need not factor through a cyclic quotient: namely, when the component is of type  $D_4$ . One easily checks that in this situation, case (1) holds.

*Remark 5.2.* We note that in both cases,  $\Psi$  is an orbit of mutually orthogonal roots.

**Lemma 5.3.** *Suppose that  $\alpha \in \Phi(G, T)$ . Then with  $\Psi$  as above, we have*

$$\sum_{\beta \in \Psi} \beta^\vee = \frac{|\Psi|}{|\Gamma \cdot \tilde{\alpha}|} \alpha^\vee.$$

*Proof.* Let  $\tilde{\alpha} \in \Phi(\tilde{G}, \tilde{T})$  be a pre-image of  $\alpha$  under the map  $i^*$  in the definition of parascopic datum. We have

$$(5.4) \quad \sum_{\beta \in \Psi} \beta = \sum_{\beta \in \Gamma \cdot \tilde{\alpha}} \beta = |\Gamma \cdot \tilde{\alpha}| \alpha.$$

Since  $\sum_{\beta \in \Psi} \beta$  is a multiple of  $\alpha$ , it follows that  $\sum_{\beta \in \Psi} \beta^\vee$  is a multiple of  $\alpha^\vee$ . To determine this multiple we let  $\beta_0 \in \Psi$ , and compute

$$\begin{aligned} \left\langle \alpha, \sum_{\beta \in \Psi} \beta^\vee \right\rangle &= \frac{1}{|\Gamma \cdot \tilde{\alpha}|} \left\langle \sum_{\beta' \in \Psi} \beta', \sum_{\beta \in \Psi} \beta^\vee \right\rangle && \text{(by (5.4))} \\ &= \frac{|\Psi|}{|\Gamma \cdot \tilde{\alpha}|} \langle \beta_0, \beta_0^\vee \rangle && \text{(by Remark 5.2)} \\ &= \frac{|\Psi|}{|\Gamma \cdot \tilde{\alpha}|} \langle \alpha, \alpha^\vee \rangle. \end{aligned}$$

Our result follows. □

**Proposition 5.5.** *There is a natural embedding  $W(G, T) \longrightarrow W(\tilde{G}, \tilde{T})^\Gamma$ . Under this map, the image of the reflection  $w_\alpha$  through the root  $\alpha \in \Phi(G, T)$  is*

$$(5.6) \quad \tilde{w} = \prod_{\beta \in \Psi} w_\beta.$$

*Proof.* We may identify  $W(G, T)$  with a subgroup of  $\text{Aut}_{\mathbb{Q}}(V^*(T))$ , which we identify with  $\text{Aut}_{\mathbb{Q}}(V^*(\tilde{T})^\Gamma)$ . To construct an embedding  $W(G, T) \longrightarrow W(\tilde{G}, \tilde{T})^\Gamma$ , it is enough to show that the image of the injective homomorphism  $W(\tilde{G}, \tilde{T})^\Gamma \longrightarrow \text{Aut}_{\mathbb{Q}}(V^*(\tilde{T})^\Gamma)$  from Corollary 3.8 is  $W(G, T)$ . Thus given  $w \in W(G, T)$ , we will show that there exists  $\tilde{w} \in W(\tilde{G}, \tilde{T})^\Gamma$  whose action on  $V^*(\tilde{T})^\Gamma$  coincides with that of  $w$ . It suffices to prove the existence of  $\tilde{w}$  only in the case in which  $w$  is a reflection  $w_\alpha$  through a root  $\alpha \in \Phi(G, T)$ . In this case, our candidate for  $\tilde{w}$  is given by (5.6).

We must show that  $\tilde{w}$  and  $w_\alpha$  have the same action on  $V^*(\tilde{T})^\Gamma$ . Let  $v \in V^*(T) = V^*(\tilde{T})^\Gamma$ . Since the roots in  $\Psi$  are orthogonal,

$$(5.7) \quad \tilde{w}(v) = v - \sum_{\beta \in \Psi} \langle v, \beta^\vee \rangle \beta,$$

where  $\beta^\vee$  denotes the coroot associated to  $\beta$ . Since the pairing  $\langle \cdot, \cdot \rangle$  is  $\Gamma$ -invariant and  $v$  is  $\Gamma$ -fixed, (5.7) is equal to

$$(5.8) \quad v - \langle v, \beta_0^\vee \rangle \sum_{\beta \in \Psi} \beta.$$

for any root  $\beta_0 \in \Psi$ . From (5.4),

$$\begin{aligned} \langle v, \beta_0^\vee \rangle \sum_{\beta \in \Psi} \beta &= |\Gamma \cdot \tilde{\alpha}| \langle v, \beta_0^\vee \rangle \alpha \\ &= \frac{|\Gamma \cdot \tilde{\alpha}|}{|\Psi|} \left\langle v, \sum_{\beta \in \Psi} \beta^\vee \right\rangle \alpha \\ &= \langle v, \alpha^\vee \rangle \alpha \quad (\text{by Lemma 5.3}). \end{aligned}$$

Hence by (5.7) and (5.8), we have that  $\tilde{w}(v) = w_\alpha(v)$ . It follows that  $\tilde{w}_\alpha = \tilde{w}$ . The proposition follows.  $\square$

## 6. STABLE CONJUGACY CLASSES OF $k$ -POINTS AND MAXIMAL $k$ -TORI

The following results might be well known, but we haven't found them in precisely the forms that we need. The arguments are standard, as exemplified by, say, the proof of [14, Lemma 6.4], or [9, §2].

Let  $G$  denote a connected reductive group over a field  $k$ . Following Kottwitz [10], we say that two elements  $s_1, s_2 \in G(k)$  are *stably conjugate* if there is some  $g \in G(k^{\text{sep}})$  such that  ${}^g s_1 = s_2$ , and for all  $\sigma \in \text{Gal}(k)$ , we have that  $g^{-1} \sigma(g) \in C_G(s_1)^\circ$ . Moreover, we say that two maximal  $k$ -tori  $T_1, T_2 \subseteq G$  are *stably conjugate* in  $G$  if there is some element  $g \in G(k^{\text{sep}})$  such that  $\text{Int}(g)$  restricts to a  $k$ -isomorphism from  $T_1$  to  $T_2$ . Let  $\mathcal{T}_{\text{st}}(G, k)$  denote the set of stable conjugacy classes of maximal  $k$ -tori in  $G$ .

**Proposition 6.1.** *Let  $s \in G(k)$  be semisimple. Let  $A_G(s)$  denote the component group of  $C_G(s)$ . Then the stable conjugacy classes in the geometric conjugacy class of  $s$  are naturally parametrized by the image  $\mathcal{C}$  of*

$$\ker[H^1(k, C_G(s)) \longrightarrow H^1(k, G)]$$

under the natural map  $H^1(k, C_G(s)) \longrightarrow H^1(k, A_G(s))$ .

*Proof.* Suppose that  $s_1$  and  $s_2$  are geometrically conjugate to  $s$ . Then we have elements  $g_i \in G(k^{\text{sep}})$  such that  ${}^{g_i}s = s_i$ . Define cocycles  $f_i \in Z^1(k, C_G(s))$  by  $f_i(\sigma) = g_i^{-1}\sigma(g_i)$  ( $\sigma \in \text{Gal}(k)$ ). Up to cohomology, these cocycles do not depend on the choices of  $g_i$ . Define  $\bar{f}_i \in Z^1(k, A_G(s))$  by setting  $\bar{f}_i(\sigma)$  to be the image of  $f_i(\sigma)$  in  $A_G(s)$ . Thus we have a map from the geometric conjugacy class of  $s$  to  $\mathcal{C}$ . It is clear that every element of  $\mathcal{C}$  arises in this way, so our map is surjective. It only remains to show that our map induces a well-defined, injective map on the set of stable conjugacy classes.

Suppose that  $s_1$  and  $s_2$  are stably conjugate. We want to show that the cocycles  $\bar{f}_1$  and  $\bar{f}_2$  are cohomologous in  $A_G(s)$ . Choose  $h \in G(k^{\text{sep}})$  such that  ${}^hs_1 = s_2$  and  $h^{-1}\sigma(h) \in C_G(s_1)^\circ$  for all  $\sigma \in \text{Gal}(k)$ . Then  $h\sigma(h)^{-1} = {}^h(h^{-1}\sigma(h))^{-1} \in C_G(s_2)^\circ$ . Let  $c = g_2^{-1}hg_1$ , which belongs to  $C_G(s)$ . Define  $f'_1 \in Z^1(k, C_G(s))$  by  $f'_1(\sigma) = c \cdot f_1(\sigma) \cdot \sigma(c)^{-1}$ , and let  $\bar{f}'_1$  be its image in  $Z^1(k, A_G(s))$ . Then by construction,  $\bar{f}'_1$  is cohomologous to  $\bar{f}_1$  in  $A_G(s)$ . It is thus enough to show that  $\bar{f}_2 = \bar{f}'_1$ . Note that

$$\begin{aligned} (f_2(\sigma))^{-1} \cdot (f'_1(\sigma)) &= (g_2^{-1}\sigma(g_2))^{-1} \cdot (g_2^{-1}h\sigma(h)^{-1}\sigma(g_2)) \\ (6.2) \quad &= (g_2^{-1}\sigma(g_2))^{-1} g_2^{-1}h\sigma(h)^{-1} g_2 (g_2^{-1}\sigma(g_2)) \\ &= \text{Int}(g_2^{-1}\sigma(g_2))^{-1} \left( \text{Int}(g_2)^{-1} (h\sigma(h)^{-1}) \right), \end{aligned}$$

which belongs to  $C_G(s)^\circ$  (as desired) since  $h\sigma(h)^{-1} \in C_G(s_2)^\circ$ .

Conversely, suppose that  $\bar{f}_1$  and  $\bar{f}_2$  are cohomologous in  $A_G(s)$ . Then there is some  $\bar{c} \in A_G(s)$  such that for all  $\sigma \in \text{Gal}(k)$ , we have  $\bar{f}_2(\sigma) = \bar{c}[\bar{f}_1(\sigma)]\sigma(\bar{c})^{-1}$ . Let  $c \in C_G(s)$  be a preimage of  $\bar{c}$ , and as above define  $f'_1 \in Z^1(k, C_G(s))$  by  $f'_1(\sigma) = c(f_1(\sigma))\sigma(c)^{-1}$ . Let  $h = g_2cg_1^{-1}$ . By construction,  $(f_2(\sigma))^{-1} \cdot f'_1(\sigma) \in C_G(s)^\circ$ . But from (6.2), this can only happen if  $h^{-1}\sigma(h) \in C_G(s_1)^\circ$ . Thus,  $s_1$  and  $s_2$  are stably conjugate.  $\square$

**Corollary 6.3.** *If  $k$  is perfect and has cohomological dimension  $\leq 1$ , then for a semisimple element in  $G(k)$ , stable conjugacy and  $G(k)$ -conjugacy coincide.*

*Proof.* From the Lang-Steinberg Theorem (see [16, §III.2.3]),  $H^1(k, L)$  is trivial for every connected reductive  $k$ -group  $L$ . (In fact, this is the only property of  $k$  that we use.)

Let  $s \in G(k)$  be semisimple. Recall that the  $G(k)$ -conjugacy classes in the geometric conjugacy class of  $s$  are parametrized by the kernel of the natural map  $H^1(k, C_G(s)) \longrightarrow H^1(k, G)$ . But this is just  $H^1(k, C_G(s))$ . Meanwhile, the natural map  $H^1(k, C_G(s)) \longrightarrow H^1(k, A_G(s))$  is injective, so our result follows from Proposition 6.1.  $\square$

**Proposition 6.4.** *Let  $S$  be a maximal  $k$ -torus of  $G$ . Then there is a natural injection  $\mathcal{T}_{\text{st}}(G, k) \longrightarrow H^1(k, W(G, S))$ . This map is a surjection when  $G$  is  $k$ -quasisplit.*

*Proof.* Let  $S$  and  $T$  be maximal  $k$ -tori in  $G$ . Then there exists  $g \in G(k^{\text{sep}})$  such that  ${}^gS = T$ . Let  $f \in Z^1(k, N_G(S))$  be the cocycle  $\sigma \mapsto g^{-1}\sigma(g)$ , and let  $\bar{f}$  be the image of  $f$  in  $Z^1(k, W(G, S))$ . Associate to  $T$  the class of  $\bar{f}$  in  $H^1(k, W(G, S))$ . This class is independent of the particular element  $g$ . The proof that this cohomology

class uniquely determines the stable conjugacy class of  $T$  and the proof of the converse of this statement are entirely analogous to the corresponding arguments in the proof of Proposition 6.1. Thus, we have the desired injection.

Suppose that  $G$  is  $k$ -quasisplit. Then  $G$  contains a maximal  $k$ -torus  $T_0$  that contains a maximal  $k$ -split torus. Applying the above paragraph to the case of  $S = T_0$ , we obtain an injection  $\mathcal{T}_{\text{st}}(G, k) \rightarrow H^1(k, W(G, T_0))$ , and this map is a surjection from [13, Thm. 1.1]. (Although this surjectivity result is stated only for semisimple groups, it is easily extended to the reductive case.)

It remains to show that we still obtain a surjection when we replace  $T_0$  by any maximal  $k$ -torus  $S$  in  $G$ . There is some element  $h \in G(k^{\text{sep}})$  such that  ${}^hT_0 = S$ . Define a map  $\phi: Z^1(k, W(G, T_0)) \rightarrow Z^1(k, W(G, S))$  as follows. If  $f$  is a cocycle for  $W(G, T_0)$  and  $\sigma \in \text{Gal}(k)$ , let  $(\phi(f))(\sigma) = hf(\sigma)\sigma(h)^{-1}$ . It is straightforward to see that  $\phi$  is a bijection of sets; that two cocycles  $f$  and  $f'$  for  $W(G, T_0)$  are cohomologous if and only if  $\phi(f)$  and  $\phi(f')$  are cohomologous; that we thus obtain a bijection (of sets, though not of pointed sets)  $H^1(k, W(G, T_0)) \rightarrow H^1(k, W(G, S))$ ; and that this bijection does not depend on the choice of  $h$ .  $\square$

*Remark 6.5.* The statement and proof of Proposition 6.4 are essentially the same as those for [14, Proposition 6.1]. We included a proof here only because our hypotheses and conclusion are slightly different.

## 7. PARASCOPY: BASIC PROPERTIES

We are now equipped to state and prove the basic properties of parascopy.

**Definition 7.1.** Let  $(\tilde{G}, \Gamma, G)$  be as in Definition 4.1, and let  $(\phi, j_*)$  and  $(\phi', j'_*)$  denote two parascopic data for  $(\tilde{G}, \Gamma, G)$ . Let  $T$  and  $T'$  denote the implicit tori for these data in  $G$ , and let  $\tilde{T}$  and  $\tilde{T}'$  denote the implicit tori for these data in  $\tilde{G}$ . We say that the parascopic data are *equivalent* if there exist elements  $g \in G(k^{\text{sep}})$  and  $\tilde{g} \in \tilde{G}(k^{\text{sep}})$  satisfying:

- (a)  ${}^gT = T'$  and  ${}^{\tilde{g}}\tilde{T} = \tilde{T}'$ .
- (b) For all  $\sigma \in \text{Gal}(k)$ ,  $i(\overline{g^{-1}\sigma(g)}) = \overline{\tilde{g}^{-1}\sigma(\tilde{g})}$ , where  $i: W(G, T) \rightarrow W(\tilde{G}, \tilde{T})^\Gamma$  is the map given in Proposition 5.5 and the bars represent the natural maps from normalizers to Weyl groups of maximal tori.
- (c) For all  $\gamma \in \Gamma$ ,  $\phi'(\gamma) = \text{Int}(\tilde{g}) \circ \phi(\gamma) \circ \text{Int}(\tilde{g})^{-1}$ .
- (d)  $j'_* = \text{Int}(\tilde{g}) \circ j_* \circ \text{Int}(g)^{-1}$ .

In this case, we will say that  $(\phi, j_*)$  is equivalent to  $(\phi', j'_*)$  *via the elements  $g$  and  $\tilde{g}$* .

It is straightforward to verify that this relation is reflexive and symmetric. To see that it is transitive, suppose that  $(\phi, j_*)$  is equivalent to  $(\phi', j'_*)$  via the elements  $g \in G(k^{\text{sep}})$  and  $\tilde{g} \in \tilde{G}(k^{\text{sep}})$ , and  $(\phi', j'_*)$  is equivalent to  $(\phi'', j''_*)$ , a parascopic datum with implicit tori  $T''$  and  $\tilde{T}''$ , via the elements  $g' \in G(k^{\text{sep}})$  and  $\tilde{g}' \in \tilde{G}(k^{\text{sep}})$ . Then  $i'(\overline{g'^{-1}\sigma(g')}) = \overline{\tilde{g}'^{-1}\sigma(\tilde{g}')}$ , where  $i': W(G, T') \rightarrow W(\tilde{G}, \tilde{T}')$  is given

by  $\text{Int}(\tilde{g}) \circ i \circ \text{Int}(g)^{-1}$ . Note that  ${}^g T = T''$  and  $\tilde{g}' \tilde{g} \tilde{T} = \tilde{T}''$ . Moreover, we have

$$\begin{aligned}
 i\left(\overline{(g'g)^{-1}\sigma(g'g)}\right) &= i\left(\overline{g^{-1}g'^{-1}\sigma(g')\sigma(g)}\right) \\
 &= i\left(\overline{g^{-1}g'^{-1}\sigma(g')gg^{-1}\sigma(g)}\right) \\
 &= i\left(\overline{g^{-1}g'^{-1}\sigma(g')g}\right) \cdot i\left(\overline{g^{-1}\sigma(g)}\right) \\
 &= \tilde{g}^{-1}i'\left(\overline{g'^{-1}\sigma(g')}\right)\tilde{g} \cdot i\left(\overline{g^{-1}\sigma(g)}\right) \\
 &= \overline{\tilde{g}^{-1}\tilde{g}'^{-1}\sigma(\tilde{g}')\tilde{g}} \cdot \overline{\tilde{g}^{-1}\sigma(\tilde{g})} \\
 &= \overline{(\tilde{g}'\tilde{g})^{-1}\sigma(\tilde{g}'\tilde{g})}.
 \end{aligned}$$

It follows that  $(\phi, j_*)$  is equivalent to  $(\phi'', j'_*)$  via the elements  $g'g \in G(k^{\text{sep}})$  and  $\tilde{g}'\tilde{g} \in \tilde{G}(k^{\text{sep}})$ .

The next result addresses the question: how large is an equivalence class of parascopic data?

**Proposition 7.2.** *Suppose  $(\phi, j_*)$  is a parascopic datum for  $(\tilde{G}, \Gamma, G)$  with implicit tori  $T \subseteq G$  and  $\tilde{T} \subseteq \tilde{G}$ .*

- (a) *Suppose  $\tilde{G}$  is  $k$ -quasisplit over. For every maximal  $k$ -torus  $T' \subseteq G$ , there exists a parascopic datum  $(\phi', j'_*)$  equivalent to  $(\phi, j_*)$  whose implicit torus in  $G$  is  $T'$ .*
- (b) *For every maximal  $k$ -torus  $\tilde{T}' \subseteq \tilde{G}$ , there exists a parascopic datum equivalent to  $(\phi, j_*)$  with implicit tori  $T$  and  $\tilde{T}'$  if and only if  $\tilde{T}'$  is stably conjugate to  $\tilde{T}$  in  $\tilde{G}$ .*
- (c) *Suppose  $(\phi', j'_*)$  is a parascopic datum with the same implicit tori as  $(\phi, j_*)$ . Then the two data are equivalent if and only if there are elements  $w \in W(G, T)$  and  $\tilde{w} \in W(\tilde{G}, \tilde{T})$  such that  $\tilde{w} \cdot i(w)^{-1} \in W(\tilde{G}, \tilde{T})^{\text{Gal}(k)}$ ,  $\phi'(\gamma) = \tilde{w} \circ \phi(\gamma) \circ \tilde{w}^{-1}$  for all  $\gamma \in \Gamma$ , and  $j'_* = \tilde{w} \circ j_* \circ w^{-1}$ .*

*Proof.* (a) Pick  $g \in G(k^{\text{sep}})$  such that  ${}^g T = T'$ . Then the function on  $\text{Gal}(k)$  given by  $f: \sigma \mapsto \overline{g^{-1}\sigma(g)}$  is a cocycle in  $Z^1(k, W(G, T))$ . Let  $\tilde{f}$  denote the image of  $f$  in  $Z^1(k, W(\tilde{G}, \tilde{T})^\Gamma)$ . Then the class of  $\tilde{f}$  in  $H^1(k, W(\tilde{G}, \tilde{T})^\Gamma)$  is independent of the choice of  $g$ . From Proposition 6.4, there is some element  $\tilde{g} \in \tilde{G}(k^{\text{sep}})$  such that for all  $\sigma \in \text{Gal}(k)$ , we have that  $\tilde{f}(\sigma) = \overline{\tilde{g}^{-1}\sigma(\tilde{g})}$ . Let  $\tilde{T}' = \tilde{g}\tilde{T}$ . Define  $\phi': \Gamma \rightarrow \text{Aut}_k(\tilde{T}')$  by  $\phi'(\gamma) = \text{Int}(\tilde{g}) \circ \phi(\gamma) \circ \text{Int}(\tilde{g})^{-1}$ , and define  $j'_*: V_*(T') \rightarrow V_*(\tilde{T}')^\Gamma$  by  $j'_* = \text{Int}(\tilde{g}) \circ j_* \circ \text{Int}(g)^{-1}$ . Then it is clear that  $(\phi', j'_*)$  is a parascopic datum for  $(\tilde{G}, \Gamma, G)$ , and it is equivalent to  $(\phi, j_*)$ .

(b) Suppose such a datum exists, equivalent to  $(\phi, j_*)$  via  $g \in N_G(T)(k^{\text{sep}})$  and  $\tilde{g} \in \tilde{G}(k^{\text{sep}})$ . We may choose  $\tilde{n} \in N_{\tilde{G}}(\tilde{T})(k^{\text{sep}})$  such that  $\tilde{n} = i(\tilde{g})^{-1}$ . Since  $i(\overline{g^{-1}\sigma(g)}) = \overline{\tilde{g}^{-1}\sigma(\tilde{g})}$ , it follows that  $(\overline{\tilde{g}\tilde{n}})^{-1}\sigma(\overline{\tilde{g}\tilde{n}}) = 1$  in  $W(\tilde{G}, \tilde{T})$  for all  $\sigma \in \text{Gal}(k)$ . Thus the map  $\text{Int}(\overline{\tilde{g}\tilde{n}}): \tilde{T} \rightarrow \tilde{T}'$  is defined over  $k$ , and so  $\tilde{T}$  and  $\tilde{T}'$  are stably conjugate. The converse is proved similarly.

(c) Suppose that the equivalence is via the elements  $g \in G(k^{\text{sep}})$  and  $\tilde{g} \in \tilde{G}(k^{\text{sep}})$ . Then  $g$  and  $\tilde{g}$  normalize  $T$  and  $\tilde{T}$ , so we have elements  $w = \overline{g}$  and  $\tilde{w} = \overline{\tilde{g}}$  in  $W(G, T)$  and  $W(\tilde{G}, \tilde{T})$ . The compatibility condition between  $g$  and  $\tilde{g}$  implies that  $\sigma(\tilde{w} \cdot i(w)^{-1}) = \tilde{w} \cdot i(w)^{-1}$  for all  $\sigma \in \text{Gal}(k)$ . The converse is proved similarly.  $\square$

## 8. DUALITY

Let  $G$  be a quasisplit reductive  $k$ -group. Let  $B_0$  denote a Borel  $k$ -subgroup of  $G$ , and  $T_0$  a maximal  $k$ -torus in  $B_0$ . Suppose that  $(G^*, B_0^*, T_0^*)$  is another triple of such groups. We say that the two triples are in  $k$ -duality if there is a  $\text{Gal}(k)$ -equivariant isomorphism  $\delta_0: \mathbf{X}^*(T_0) \rightarrow \mathbf{X}_*(T_0^*)$  that induces an isomorphism from the based root datum of  $(G, B_0, T_0)$  to that of the dual of  $(G^*, B_0^*, T_0^*)$ ; that is,

- $\delta_0$  maps the simple roots in  $\Phi(G, T_0)$  with respect to  $B_0$  onto the simple coroots in  $\Phi^\vee(G^*, T_0^*)$  with respect to  $B_0^*$ .
- The transpose  $\delta_0^*: \mathbf{X}^*(T_0^*) \rightarrow \mathbf{X}_*(T_0)$  of  $\delta_0$  maps the simple roots in  $\Phi(G^*, T_0^*)$  with respect to  $B_0^*$  onto the simple coroots in  $\Phi^\vee(G, T_0)$  with respect to  $B_0$ .

Given a triple  $(G, B_0, T_0)$ , a triple  $(G^*, B_0^*, T_0^*)$  always exists, and is unique up to  $k$ -isomorphism. In this situation, we will say that  $G^*$  is the  $k$ -dual of  $G$ . We will say that a pair of maximal  $k$ -tori  $T \subseteq G$  and  $T^* \subseteq G^*$  are in  $k$ -duality if there is a  $\text{Gal}(k)$ -equivariant isomorphism  $\delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*)$  such that

$$(8.1) \quad \delta(\Phi(G, T)) = \Phi^\vee(G^*, T^*) \quad \text{and} \quad \delta^*(\Phi(G^*, T^*)) = \Phi^\vee(G, T),$$

where  $\delta^*: \mathbf{X}^*(T^*) \rightarrow \mathbf{X}_*(T)$  is the transpose of  $\delta$ . (Note that this notion of duality of tori depends on the ambient groups  $G$  and  $G^*$ .) The isomorphism  $\delta$  will be referred to as a *duality map*.

*Remark 8.2.* Note that a duality map  $\delta$  determines a  $\text{Gal}(k)$ -equivariant isomorphism, also denoted  $\delta$ , from  $W(G, T)$  to  $W(G^*, T^*)$  under which, for each  $\alpha \in \Phi(G, T)$ , the reflection  $w_\alpha$  is sent to  $w_{\delta\alpha}$ . Then for every  $w \in W(G, T)$  and every  $\chi \in \mathbf{X}^*(T)$ ,  ${}^w\chi = {}^{\delta(w)}\delta(\chi)$ . In other words, if we identify  $W(G, T)$  and  $W(G^*, T^*)$  via  $w_\alpha \longleftrightarrow w_{\delta\alpha}$ , then  $\delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*)$  is a  $W(G, T)$ -equivariant map.

We note that when  $k$  is a finite field, it is standard (see [6–8]) to work with an *anti-action* of  $W(G, T)$  on  $\mathbf{X}^*(T)$  (satisfying  ${}^{w_1 w_2}\chi = {}^{w_2}({}^{w_1}\chi)$ ). This, in turn, makes the natural map  $W(G, T) \rightarrow W(G^*, T^*)$  an *anti-isomorphism* and forces the geometric Frobenius element  $F$  to act on the Weyl group  $W(G^*, T^*)$  via the *inverse* of its usual action on  $W(G, T)$ . However, we consider the standard action of  $W(G, T)$  on  $\mathbf{X}^*(T)$ , which results in the isomorphism  $\delta$  of Weyl groups in the preceding paragraph. Using this map to identify  $W(G, T)$  and  $W(G^*, T^*)$ , one sees easily that  $\text{Gal}(k)$  acts in the same way on these groups.

**Proposition 8.3.** *There is a canonical one-to-one correspondence  $\mathcal{T}_{\text{st}}(G, k) \longleftrightarrow \mathcal{T}_{\text{st}}(G^*, k)$ . If  $T \subseteq G$  and  $T^* \subseteq G^*$  correspond, then they are in  $k$ -duality, and the duality map  $\delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*)$  is uniquely determined up to the action of  $W(G, T)^{\text{Gal}(k)}$ .*

*Proof.* Fix maximal  $k$ -tori  $T_0 \subseteq G$  and  $T_0^* \subseteq G^*$  and a duality map  $\delta_0: \mathbf{X}^*(T_0) \rightarrow \mathbf{X}_*(T_0^*)$  of based root data as in the definition of  $k$ -duality above. As described in Proposition 6.4, we have a bijection between the set of stable conjugacy classes of maximal  $k$ -tori of  $G$  (resp.  $G^*$ ) and  $H^1(k, W(G, T_0))$  (resp.  $H^1(k, W(G^*, T_0^*))$ ). Since  $\delta_0: W(G, T_0) \rightarrow W(G^*, T_0^*)$  induces isomorphisms  $Z^1(k, W(G, T_0)) \rightarrow Z^1(k, W(G^*, T_0^*))$  and  $H^1(k, W(G, T_0)) \rightarrow H^1(k, W(G^*, T_0^*))$  on the corresponding sets of cocycles and cohomology classes, we have the desired correspondence between the sets of stable conjugacy classes of maximal  $k$ -tori in  $G$  and  $G^*$ .

Let  $T \subseteq G$  and  $T^* \subseteq G^*$  be maximal  $k$ -tori whose stable conjugacy classes correspond as in the preceding paragraph. Then  $T = {}^g T_0$  for some  $g \in G(k^{\text{sep}})$ ,

so  $T$  can be obtained from  $T_0$  by twisting by the cocycle  $f(\sigma) = g^{-1}\sigma(g)$ . Let  $f^*$  denote the image of  $f$  in  $Z^1(k, W(G^*, T_0^*))$  under the correspondence induced by  $\delta_0$  above. From Proposition 6.4, there is some  $h^* \in G^*(k^{\text{sep}})$  such that  $f^*$  is cohomologous to the cocycle that takes each  $\sigma \in \text{Gal}(k)$  to the image in  $W(G^*, T_0^*)$  of  $h^{*-1}\sigma(h^*)$ . That is, there is some  $w^* \in W(G^*, T_0^*)$  such that for all  $\sigma \in \text{Gal}(k)$ , we have  $f^*(\sigma) = w^{*-1}h^{*-1}\sigma(h^*)\sigma(w^*)$ . Let  $g^* = h^*n^*$ , where  $n^*$  is a pre-image in  $N_{G^*}(T_0^*)$  of  $w^*$ . Then  $T^* = g^*T_0^*$ , and  $f^*$  takes each  $\sigma$  to the image in  $W(G^*, T_0^*)$  of  $g^{*-1}\sigma(g^*)$ .

To show that  $T$  and  $T^*$  are in  $k$ -duality, define an isomorphism  $\delta: \mathbf{X}^*(T) \rightarrow \mathbf{X}_*(T^*)$  as follows. An element of  $\mathbf{X}^*(T)$  can be written in the form  ${}^g\chi$  for a unique  $\chi \in \mathbf{X}^*(T_0)$ . Let

$$(8.4) \quad \delta({}^g\chi) = g^*(\delta_0\chi).$$

It is easily verified that  $\delta$  satisfies (8.1).

We need to show that  $\delta$  is  $\text{Gal}(k)$ -equivariant. Suppose  $\sigma \in \text{Gal}(k)$ . Then

$$\begin{aligned} \delta(\sigma({}^g\chi)) &= \delta(\sigma^{(g)}(\sigma\chi)) \\ &= \delta(gg^{-1}\sigma(g)(\sigma\chi)) \\ &= g^*(\delta_0(g^{-1}\sigma(g)(\sigma\chi))) \\ &= g^*(\delta_0(f(\sigma)(\sigma\chi))). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma(\delta({}^g\chi)) &= \sigma(g^*(\delta_0\chi)) \\ &= \sigma(g^*)(\sigma(\delta_0\chi)) \\ &= \sigma(g^*)(\delta_0(\sigma\chi)) \end{aligned}$$

It follows that  $\delta$  will be  $\text{Gal}(k)$ -equivariant if and only if

$$\delta_0(f(\sigma)(\sigma\chi)) = g^{*-1}\sigma(g^*)(\delta_0(\sigma\chi)) = f^*(\sigma)(\delta_0(\sigma\chi))$$

for all  $\sigma \in \text{Gal}(k)$  and  $\chi \in \mathbf{X}^*(T_0)$ . But this equality follows from the equivariance of  $\delta_0$  with respect to the action of  $W(G, T_0) = W(G^*, T_0^*)$  (see Remark 8.2) and the fact that  $f^*(\sigma) = \delta_0(f(\sigma))$ . Thus,  $\delta$  is a duality map. Finally, note that  $\delta$  depends only up to conjugacy by an element of  $W(G, T)^{\text{Gal}(k)}$  on the choices of the elements  $g$  and  $g^*$ .  $\square$

*Remark 8.5.* Given a maximal  $k$ -torus  $S \subseteq G$ , let  $S^* \subseteq G^*$  correspond to  $S$  via Proposition 8.3. Repeating the proof with  $S$  and  $S^*$  in place of  $T_0$  and  $T_0^*$ , we see the following. If the maximal  $k$ -torus  $T \subseteq G$  corresponds to  $T^* \subseteq G^*$ , and we write  $T = {}^gS$  with  $g \in G(k^{\text{sep}})$ , then there exists  $g^* \in G^*(k^{\text{sep}})$  such that  $T^* = g^*S^*$ , and the cocycle that sends  $\sigma \in \text{Gal}(k)$  to the image in  $W(G, S)$  of  $g^{-1}\sigma(g)$  corresponds to the analogous cocycle determined by  $g^*$  under the identification between  $W(G, S)$  and  $W(G^*, S^*)$ . Thus, this latter identification leads to the same correspondence  $\mathcal{T}_{\text{st}}(G, k) \longleftrightarrow \mathcal{T}_{\text{st}}(G^*, k)$  of Proposition 8.3. Moreover, if  $\delta_S$  and  $\delta_T$  are duality maps for  $S$  and  $T$ , then there is some  $w \in W(G, S)^{\text{Gal}(k)}$  such for all  $\chi \in \mathbf{X}^*(S)$ , we have  $g^*w\delta_S(\chi) = \delta_T({}^g\chi)$ . Since  $\chi \mapsto w\delta_S(\chi)$  is another duality map for  $S$ , we may replace  $\delta_S$  by it and then we can write  $g^*\delta_S(\chi) = \delta_T({}^g\chi)$  in analogy with (8.4).



## 9. THE CONORM MAP

From now on, let  $\tilde{G}$  and  $G$  be  $k$ -quasisplit connected reductive  $k$ -groups,  $\Gamma$  a finite group, and  $(\phi, j_*)$  a parascopic datum for  $(\tilde{G}, \Gamma, G)$  with implicit tori  $T$  and  $\tilde{T}$ .

*Remark 9.1.* Given a maximal  $k$ -torus  $S \subseteq G$ , one has from Proposition 7.2(a,b) a torus  $\tilde{S} \subseteq \tilde{G}$ , uniquely determined up to stable conjugacy, such that there is a parascopic datum equivalent to  $(\phi, j_*)$  having implicit tori  $S$  and  $\tilde{S}$ . By Proposition 8.3,  $S$  and  $\tilde{S}$  determine maximal  $k$ -tori  $S^* \subseteq G^*$  and  $\tilde{S}^* \subseteq \tilde{G}^*$  (up to stable conjugacy), and duality maps and duality maps  $\delta: \mathbf{X}^*(S) \rightarrow \mathbf{X}_*(S^*)$  and  $\tilde{\delta}: \mathbf{X}^*(\tilde{S}) \rightarrow \mathbf{X}_*(\tilde{S}^*)$  (up to conjugacy by  $W(G, S)^{\text{Gal}(k)}$  and  $W(\tilde{G}, \tilde{S})^{\text{Gal}(k)}$ , respectively). Similarly, given a maximal  $k$ -torus  $S^* \subseteq G^*$ , one obtains  $S \subseteq G$  and thus all of the other data above.

Using  $T$  and  $\tilde{T}$ , Choose  $T^*$ ,  $\tilde{T}^*$ ,  $\delta$ , and  $\tilde{\delta}$  as in the Remark. From §2 and Condition P1, the parascopic datum  $(\phi, j_*)$  determines a map  $\mathcal{N}^* = \mathcal{N}_T^*: \mathbf{X}^*(T) \rightarrow \mathbf{X}^*(\tilde{T})$ . Define

$$\hat{\mathcal{N}}_{T^*,*} := \tilde{\delta} \circ \mathcal{N}^* \circ \delta^{-1}: \mathbf{X}_*(T^*) \rightarrow \mathbf{X}_*(\tilde{T}^*).$$

Then  $\hat{\mathcal{N}}_{T^*,*}$  determines a *conorm* homomorphism

$$\hat{\mathcal{N}}_{T^*}: T^* \rightarrow \tilde{T}^*.$$

Since  $\Gamma$  consists of  $k$ -automorphisms,  $\mathcal{N}^*$  is  $\text{Gal}(k)$ -equivariant. Since both  $\delta$  and  $\tilde{\delta}$  are  $\text{Gal}(k)$ -equivariant, so is  $\hat{\mathcal{N}}_{T^*,*}$ . Hence  $\hat{\mathcal{N}}_{T^*}$  is defined over  $k$ . We also have a corresponding map  $\hat{\mathcal{N}}_{T^*}^*: \mathbf{X}^*(\tilde{T}^*) \rightarrow \mathbf{X}^*(T^*)$ . More explicitly,

$$\hat{\mathcal{N}}_{T^*}^* = \delta^* \circ \mathcal{N}_* \circ \tilde{\delta}^{*-1}.$$

From Remark 8.2, the duality maps  $\delta$  and  $\tilde{\delta}$  determine identifications  $W(G, T) \rightarrow W(G^*, T^*)$  and  $W(\tilde{G}, \tilde{T}) \rightarrow W(\tilde{G}^*, \tilde{T}^*)$ . Thus, the embedding  $i$  of  $W(G, T)$  in  $W(\tilde{G}, \tilde{T})$  (see §5) determines an embedding (which we will also denote by  $i$ ) of  $W(G^*, T^*)$  in  $W(\tilde{G}^*, \tilde{T}^*)$ .

*Remark 9.2.* If we identify  $W(G^*, T^*) = W(G, T)$  with its image in  $W(\tilde{G}^*, \tilde{T}^*) = W(\tilde{G}, \tilde{T})$  under  $i$ , it is clear that  $\mathcal{N}^*$  is  $W(G, T)$ -equivariant. Since  $\delta$  and  $\tilde{\delta}$  are  $W(G, T)$ -equivariant (given the identifications in the preceding paragraph), it follows that  $\hat{\mathcal{N}}_{T^*}$  is as well.

Let  $s \in T^*(\bar{k})$  and let  $\tilde{s} = \hat{\mathcal{N}}_{T^*}(s) \in \tilde{T}^*(\bar{k})$ . Let  $H^* = C_{G^*}(s)$  and  $\tilde{H}^* = C_{\tilde{G}^*}(\tilde{s})$ .

**Proposition 9.3.** *The embedding  $i: W(G^*, T^*) \rightarrow W(\tilde{G}^*, \tilde{T}^*)$  restricts to give embeddings of  $W(H^*, T^*)$  in  $W(\tilde{H}^*, \tilde{T}^*)$  and of  $W(H^{*\circ}, T^*)$  in  $W(\tilde{H}^{*\circ}, \tilde{T}^*)$ .*

*Proof.* Suppose that  $w \in W(H^*, T^*)$ . Then by Remark 9.2,

$$(i(w))(\tilde{s}) = (i(w))(\hat{\mathcal{N}}_{T^*}(s)) = \hat{\mathcal{N}}_{T^*}(w(s)) = \hat{\mathcal{N}}_{T^*}(s) = \tilde{s}.$$

Thus  $i$  gives an embedding of  $W(H^*, T^*)$  in  $W(\tilde{H}^*, \tilde{T}^*)$ .

Now suppose  $w \in W(H^{*\circ}, T^*)$ . According to [6, Theorem 3.5.3],  $w$  is a product of reflections through roots  $\alpha^* \in \Phi(G^*, T^*)$  such that  $\alpha^*(s) = 1$ . For such a root  $\alpha^*$ , let  $\alpha$  be the corresponding root in  $\Phi(G, T)$ :  $\alpha = \delta^{-1}(\alpha^{*\vee})$ . Then

$$i(w_{\alpha^*}) = i(w_{\alpha}) = \prod_{\beta \in \Psi} w_{\beta}$$

in the notation of Proposition 5.5. For each  $\beta \in \Psi$ , let  $\beta^*$  be the corresponding root in  $\Phi(\tilde{G}^*, \tilde{T}^*)$ :  $\beta^* = \tilde{\delta}^{*-1}(\beta^\vee)$ . Then  $i(w_{\alpha^*}) = \prod_{\Psi} w_{\beta^*}$ , and by *loc. cit.*, to show that  $i(w)$  lies in  $W(\tilde{H}^{*\circ}, \tilde{T}^*)$ , it suffices to show that  $\beta^*(\tilde{s}) = 1$  for all  $\beta \in \Psi$ . For such a root  $\beta^*$ ,

$$\beta^*(\tilde{s}) = \beta^*(\hat{\mathcal{N}}_{T^*}(s)) = (\hat{\mathcal{N}}_{T^*}^* \beta^*)(s),$$

so it is enough to show that  $\hat{\mathcal{N}}_{T^*}^* \beta^*$  is an integer multiple of  $\alpha^*$ .

Recall the identifications of  $V_*(T)$  with  $V_*(\tilde{T})^\Gamma$  and  $V^*(T)$  with  $V^*(\tilde{T})^\Gamma$  described in §5. We have

$$\begin{aligned} \mathcal{N}_{T,*} \beta^\vee &= \sum_{\gamma \in \Gamma} \gamma \cdot \beta^\vee && \text{(by (2.3))} \\ &= |\text{stab}_\Gamma \beta| \sum_{\beta' \in \Psi} (\beta')^\vee && \text{(by Remark 5.2)} \\ &= \frac{|\Psi| |\text{stab}_\Gamma \beta|}{|\Gamma \cdot \tilde{\alpha}|} \alpha^\vee && \text{(by Lemma 5.3).} \end{aligned}$$

But the constant  $|\Psi| |\text{stab}_\Gamma \beta| / |\Gamma \cdot \tilde{\alpha}|$  is always integral. Indeed, (in the terminology of §5) in case (1), we have  $\Psi = \Gamma \cdot \tilde{\alpha}$ , while in case (2),  $|\Gamma \cdot \tilde{\alpha}| = 2|\Psi|$  and  $|\text{stab}_\Gamma \beta|$  is even. Translating this to the dual setting, we have that  $\hat{\mathcal{N}}^* \beta^*$  is an integer multiple of  $\alpha^*$ , and the proposition follows.  $\square$

## 10. LIFTING OF SEMISIMPLE GEOMETRIC CONJUGACY CLASSES

We now prove Statement (A) from the Introduction.

**Proposition 10.1.** *Fix a parascopic datum for  $(\tilde{G}, \Gamma, G)$ . Then there is a canonical  $k$ -morphism from the  $k$ -variety of geometric semisimple conjugacy classes in  $G^*$  to the analogous variety for  $\tilde{G}^*$ . Moreover, if  $T^*$  is a maximal  $k$ -torus of  $G^*$  and  $s \in T^*(\bar{k})$ , then*

$$\hat{\mathcal{N}}_{T^*}(s) \in \hat{\mathcal{N}}([s]),$$

where  $[s]$  is the geometric conjugacy class of  $s$  in  $G^*$ . That is,  $\hat{\mathcal{N}}$  is compatible with the conorms  $\hat{\mathcal{N}}_{T^*}$  on all maximal  $k$ -tori.

*Proof.* Let  $S^*$  be a maximal  $k$ -torus in  $G^*$ . As in Remark 9.1, we obtain maximal  $k$ -tori  $S \subseteq G$ ,  $\tilde{S} \subseteq \tilde{G}$ , and  $\tilde{S}^* \subseteq \tilde{G}^*$ , and duality maps  $\delta_S$  and  $\delta_{\tilde{S}}$ , and thus a  $W(G^*, S^*)$ -equivariant  $k$ -morphism  $\hat{\mathcal{N}}_{S^*}: S^* \rightarrow \tilde{S}^*$ . Hence we obtain a  $k$ -morphism

$$\hat{\mathcal{N}}: S^*/W(G^*, S^*) \rightarrow \tilde{S}^*/W(\tilde{G}^*, \tilde{S}^*).$$

But these latter two varieties are  $k$ -isomorphic to the varieties of geometric semisimple conjugacy classes in  $G^*$  and  $\tilde{G}^*$ , respectively.

We now show that  $\hat{\mathcal{N}}$  is independent all of the choices involved in its definition. Let  $T^*$  be a maximal  $k$ -torus of  $G^*$ . Then from Remark 9.1 again, we have the tori  $T \subseteq G$ ,  $\tilde{T} \subseteq \tilde{G}$ , and  $\tilde{T}^* \subseteq \tilde{G}^*$ . By definition,  $T$  and  $\tilde{T}$  are the implicit tori in a parascopic datum for  $(\tilde{G}, \Gamma, G)$ , equivalent to the parascopic datum involving  $S$  and  $\tilde{S}$ . In particular, there exist elements  $g \in G$  and  $\tilde{g} \in \tilde{G}$  such that (c) and (d) in Definition 7.1 hold. It follows from this and the definition of  $\mathcal{N}^*$  that

$$(10.2) \quad \mathcal{N}_{T^*}^* = \text{Int}(\tilde{g}^{-1})^* \circ \mathcal{N}_{S^*}^* \circ \text{Int}(g)^*.$$

We may choose  $\tilde{g}^* \in \tilde{G}^*$  such that  $\tilde{g}^* \tilde{S}^* = \tilde{T}^*$  again as in Remark 8.5. Then there exist duality maps  $\delta_T, \delta_{\tilde{T}}$  such that the top and bottom faces of the following diagram commute:

$$\begin{array}{ccccc}
 & & \mathbf{X}^*(\tilde{T}) & \xrightarrow{\delta_{\tilde{T}}} & \mathbf{X}_*(\tilde{T}^*) \\
 & \nearrow \text{Int}(\tilde{g}^{-1})^* & \uparrow & & \nearrow \text{Int}(\tilde{g}^*)_* \\
 \mathbf{X}^*(\tilde{S}) & \xrightarrow{\delta_{\tilde{S}}} & \mathbf{X}_*(\tilde{S}^*) & & \uparrow \hat{\mathcal{N}}_{T^*,*} \\
 & \downarrow \mathcal{N}_T^* & \uparrow \hat{\mathcal{N}}_{S^*,*} & & \\
 & \mathbf{X}^*(T) & \xrightarrow{\delta_T} & \mathbf{X}_*(T^*) & \\
 & \nearrow \text{Int}(g^{-1})^* & \downarrow \text{Int}(g^*)_* & & \\
 \mathbf{X}^*(S) & \xrightarrow{\delta_S} & \mathbf{X}_*(S^*) & & 
 \end{array}$$

The front and back faces also commute by the definitions of  $\hat{\mathcal{N}}_{S^*,*}$  and  $\hat{\mathcal{N}}_{T^*,*}$ . The left face commutes by (10.2). Since all of the horizontal maps are isomorphisms, the right face must also commute. That is, we have the equality

$$\hat{\mathcal{N}}_{S^*,*} = \text{Int}(\tilde{g}^*)_*^{-1} \circ \hat{\mathcal{N}}_{T^*,*} \circ \text{Int}(g^*)_*$$

of maps  $\mathbf{X}_*(S^*) \rightarrow \mathbf{X}_*(\tilde{S}^*)$ . Therefore, the corresponding homomorphisms  $S^* \rightarrow \tilde{S}^*$  must be equal, i.e.,

$$(10.3) \quad \hat{\mathcal{N}}_{S^*} = \text{Int}(\tilde{g}^*)^{-1} \circ \hat{\mathcal{N}}_{T^*} \circ \text{Int}(g^*).$$

It follows that the definition of  $\hat{\mathcal{N}}$  is independent of the particular choice of torus  $S^*$ .  $\square$

## 11. MAIN THEOREM

We keep the same hypotheses on  $\tilde{G}$ ,  $G$ , and  $\Gamma$ , and we fix the same parascopic datum for  $(\tilde{G}, \Gamma, G)$  as in §10.

**Theorem 11.1.** *Let  $s_1, s_2 \in G^*(k)$  be semisimple elements that are stably conjugate, and let  $T_i^*$  be maximal  $k$ -tori in  $G^*$  containing  $s_i$ . Then the elements  $\hat{\mathcal{N}}_{T_i^*}(s_i) \in \tilde{G}^*(k)$  are stably conjugate.*

*Remark 11.2.* That is, there is a well-defined map  $\hat{\mathcal{N}}^{\text{st}}$  from the set of semisimple stable conjugacy classes in  $G^*(k)$  to the set of such classes in  $\tilde{G}^*(k)$ , such that for every semisimple  $s \in G^*(k)$  and every maximal  $k$ -torus  $T^*$  in  $G$  containing  $s$ , we have that  $\hat{\mathcal{N}}_{T^*}(s) \in \hat{\mathcal{N}}^{\text{st}}(s)$ .

*Proof of Theorem 11.1.* Define  $T_i$ ,  $\delta_i$ ,  $\tilde{T}_i$ ,  $\tilde{T}_i^*$ , and  $\tilde{\delta}_i$  as in Remark 9.1. Let  $\tilde{s}_i = \hat{\mathcal{N}}_{T_i^*}(s_i) \in \tilde{G}^*(k)$ . According to Proposition 10.1,  $\tilde{s}_1$  is geometrically conjugate to  $\tilde{s}_2$  in  $\tilde{G}^*$ . We want to show that the stable conjugacy classes of  $\tilde{s}_1$  and  $\tilde{s}_2$  in  $\tilde{G}^*(k)$  coincide.

We have that  ${}^g s_1 = s_2$  for some  $g^* \in G^*(k^{\text{sep}})$  such that  $g^{*-1} \sigma(g^*) \in H_1^{*\circ} := C_{G^*}(s_1)^\circ$  for all  $\sigma \in \text{Gal}(k)$ . Moreover, by replacing  $g^*$  by  $g^* h^*$  for an appropriate element  $h^* \in H_1^{*\circ}$ , we may assume, in addition, that  ${}^g T_1^* = T_2^*$ .

The proof of Proposition 10.1 shows that we can choose  $\tilde{g}^* \in \tilde{G}^*$  so that  $\tilde{g}^* \tilde{s}_1 = \tilde{s}_2$ ,  $\tilde{g}^* \tilde{T}_1^* = \tilde{T}_2^*$ , and the respective images of the cocycles  $\sigma \mapsto g^{*-1}\sigma(g^*)$  and  $\sigma \mapsto \tilde{g}^{*-1}\sigma(\tilde{g}^*)$  in  $Z^1(k, W(G^*, T_1^*))$  and  $Z^1(k, W(\tilde{G}^*, \tilde{T}_1^*))$  coincide under the embedding

$$Z^1(k, W(G^*, T_1^*)) \longrightarrow Z^1(k, W(\tilde{G}^*, \tilde{T}_1^*))$$

induced by  $i$ . Since  $g^{*-1}\sigma(g^*) \in H_1^{*\circ} \cap N_{G^*}(T_1^*) = N_{H_1^{*\circ}}(T_1^*)$  for all  $\sigma \in \text{Gal}(k)$ , it follows that the image of  $\tilde{g}^{*-1}\sigma(\tilde{g}^*)$  in  $W(\tilde{G}^*, \tilde{T}_1^*)$  must actually lie in  $i(W(H_1^{*\circ}, T_1^*))$ , which is contained in  $W(\tilde{H}_1^{*\circ}, \tilde{T}_1^*)$  by Proposition 9.3. Thus  $\tilde{g}^{*-1}\sigma(\tilde{g}^*) \in \tilde{H}_1^{*\circ}$  so  $\tilde{s}_1$  and  $\tilde{s}_2$  are geometrically conjugate.  $\square$

**Corollary 11.3.** *In the situation of Theorem 11.1 and Remark 11.2, suppose that  $k$  is perfect and has cohomological dimension  $\leq 1$ . Then  $\hat{N}$  refines to a map  $\hat{N}^{st}$  from the set of semisimple,  $G^*(k)$ -conjugacy classes in  $G^*(k)$  to the set of semisimple,  $\tilde{G}^*(k)$ -conjugacy classes in  $\tilde{G}^*(k)$ .*

Note that our standing hypothesis that  $\tilde{G}$  and  $G$  are  $k$ -quasisplit becomes automatic in this case.

*Proof.* This follows from Theorem 11.1 and Corollary 6.3.  $\square$

## 12. APPENDIX: WHEN IS A QUASI-SEMISIMPLE AUTOMORPHISM SEPARABLY QUASI-SEMISIMPLE?

We now justify Remark 3.2.

**Lemma 12.1.** *Suppose  $\tilde{G}$  is a connected reductive  $k$ -group, and  $\gamma$  is a quasi-semisimple  $k$ -automorphism of  $\tilde{G}$ . Suppose that at least one of the following holds:*

- (1) *the characteristic of  $k$  is not two;*
- (2)  *$k$  is perfect;*
- (3) *no power of  $\gamma$  acts via a non-inner automorphism on any factor of  $\tilde{G}$  of type  $A_{2n}$ .*

*Then  $\gamma$  is separably quasi-semisimple.*

*Proof.* Note that this statement is obvious when  $k$  is perfect.

We may assume that  $k = k^{\text{sep}}$  is separably closed. It is enough to show that  $\tilde{G}$  has a  $\gamma$ -stable maximal  $k$ -torus  $\tilde{T}_\bullet$  contained in a  $\gamma$ -stable Borel subgroup  $\tilde{B}_\bullet$ . For then  $\tilde{T}_\bullet$  must be split over  $k$ , so the root groups corresponding to the roots in  $\Phi(\tilde{G}, \tilde{T}_\bullet)$  must be defined over  $k$ , and hence so must  $\tilde{B}_\bullet$ .

Fix a Borel  $\bar{k}$ -subgroup  $\tilde{B}_*$  of  $\tilde{G}$  and a maximal  $\bar{k}$ -torus  $\tilde{T}_* \subseteq \tilde{B}_*$ . The homogeneous space  $\tilde{G}/\tilde{T}_*$  can be viewed, via the map  $g\tilde{T}_* \mapsto ({}^g\tilde{B}_*, {}^g\tilde{T}_*)$ , as the  $\bar{k}$ -variety  $X$  of all pairs  $(\tilde{B}, \tilde{T})$ , where  $\tilde{B}$  is a Borel  $\bar{k}$ -subgroup of  $\tilde{G}$ , and  $\tilde{T}$  is a maximal  $\bar{k}$ -torus of  $\tilde{B}$ . Taking  $\tilde{T}_*$  to be defined over  $k$  shows that  $X$  can be given the structure of a  $k$ -variety; this structure is easily seen to be independent of the particular choice of  $(\tilde{B}_*, \tilde{T}_*)$ .

Assume for now that  $\tilde{T}_*$  is defined over  $k$ . There is an obvious action of  $\gamma$  on  $X$ . Let  $X^\gamma$  be the (nonempty)  $\bar{k}$ -variety of  $\gamma$ -fixed points in  $X$ . A point  $x \in X^\gamma(\bar{k})$  corresponds to a pair  $(\tilde{B}, \tilde{T})$  as above such that  $\tilde{B}$  and  $\tilde{T}$  are  $\gamma$ -stable. Moreover, if  $x \in X(k) \cap X^\gamma$  is represented by  $g \in \tilde{G}(\bar{k})$ , then  $\text{Int}(g) : \tilde{T}_* \rightarrow \tilde{T}$  is a  $k$ -isomorphism. Thus  $\tilde{T}$  (and hence  $\tilde{B}$ ) are defined over  $k$  [5, Cor. 14.5]. It follows

that the desired Borel-torus pair exists provided that  $X(k) \cap X^\gamma \neq \emptyset$ . To prove this nonemptiness, it suffices to show that  $X^\gamma$  is defined over  $k$ , for then  $X^\gamma(k)$  must be dense in  $X^\gamma$  [5, Cor. 13.3]. By [17, Thm. 11.2.13], this will follow in turn if we can prove the equality of tangent spaces

$$(12.2) \quad T_x(X^\gamma) = T_x(X)^\gamma \quad \text{for all } x \in X^\gamma(\bar{k}).$$

Let  $\tilde{T}_\bullet$  and  $\tilde{B}_\bullet$  be as in Definition 3.1. The Bruhat decomposition can be used to express  $\tilde{G}$  as a disjoint union

$$(12.3) \quad \coprod_{w \in W(\tilde{G}, \tilde{T}_\bullet)} \tilde{U} \tilde{U}_w n_w \tilde{T}_\bullet,$$

where  $n_w$  is a representative of  $w$  in  $N_{\tilde{G}}(\tilde{T}_\bullet)$ ,  $\tilde{U}$  is the unipotent radical of  $\tilde{B}_\bullet$ , and  $\tilde{U}_w$  is the group generated by the root groups corresponding to  $\alpha \in \Phi^-(\tilde{G}, \tilde{T}_\bullet) \cap w\Phi^+(\tilde{G}, \tilde{T}_\bullet)$ , where positivity is with respect to  $\tilde{B}_\bullet$ . Moreover, for each  $w$ , the obvious map  $\tilde{U} \times \tilde{U}_w \times \tilde{T}_\bullet \rightarrow \tilde{U} \tilde{U}_w n_w \tilde{T}_\bullet$  is an isomorphism of varieties.

Let  $z \in X(\bar{k}) = (\tilde{G}/\tilde{T}_\bullet)(\bar{k})$  be the point corresponding to the coset  $\tilde{T}_\bullet$ . It follows easily from (12.3) that

$$(12.4) \quad (\tilde{U} \tilde{U}_w n_w \tilde{T}_\bullet \cdot z)^\gamma = \begin{cases} \tilde{U}^\gamma \tilde{U}_w^\gamma n_w \cdot z & \text{if } w \in W(\tilde{G}, \tilde{T}_\bullet)^\gamma, \\ \emptyset & \text{otherwise.} \end{cases}$$

From Lemma 3.3,  $G = (\tilde{G}^\gamma)^\circ$  is a reductive group,  $T_\bullet := (\tilde{T}_\bullet^\gamma)^\circ = G \cap \tilde{T}_\bullet$  is a maximal torus in  $G$ , and  $B_\bullet := \tilde{B}_\bullet^\gamma$  is a Borel subgroup of  $G$  containing  $T_\bullet$ .

Observe that  $G$  acts on  $X^\gamma$  via left translation. It follows from (12.4) that  $X^\gamma$  is the union of a finite number of  $G$ -orbits represented by some subset of  $\{n_w\}$ . (In fact, we show in Proposition 3.5 that  $W(G, T_\bullet)$  embeds in  $W(\tilde{G}, \tilde{T}_\bullet)^\gamma$ , implying that one can take as representatives for  $G \backslash X^\gamma$  any subset of  $\{n_w\}$  corresponding to representatives of  $W(G, T_\bullet) \backslash W(\tilde{G}, \tilde{T}_\bullet)^\gamma$  in  $W(\tilde{G}, \tilde{T}_\bullet)^\gamma$ .) Moreover, the stabilizer in  $G$  of  $n_w \cdot z$  is precisely  $T_\bullet$ . In particular, each  $G$ -orbit has dimension  $\dim G - \dim T_\bullet$ . Since these orbits are irreducible and of the same dimension as  $X^\gamma$ , it is straightforward to show that they are precisely the irreducible components of  $X^\gamma$ . Since they are disjoint and finite in number, they must also be the connected components of  $X^\gamma$ . It therefore suffices to verify (12.2) for  $x = n_w \cdot z$ .

Since the dimension of any  $G$ -orbit in  $X^\gamma$  is  $\dim G - \dim T_\bullet$ , we have  $\dim T_x(X^\gamma) = \dim T_x(G \cdot x) \geq \dim G - \dim T_\bullet$ . On the other hand, there is a natural identification  $T_x(X) = L(\tilde{G})/L(\tilde{T}_\bullet)$ , where  $L(\ )$  denotes the taking of a Lie algebra, so we have

$$T_x(X^\gamma) \subseteq T_x(X)^\gamma = \left( L(\tilde{G})/L(\tilde{T}_\bullet) \right)^\gamma = \left( \bigoplus_{\alpha \in \Phi(\tilde{G}, \tilde{T}_\bullet)} L(\tilde{U}_\alpha) \right)^\gamma.$$

It follows from [18, §8.2(2''')] that if either assumption (1) or (3) holds, the last space is equal to

$$\bigoplus_{\beta \in \Phi(G, T_\bullet)} L(U_\beta),$$

which has dimension  $|\Phi(G, T_\bullet)| = \dim G - \dim T_\bullet$ . Thus  $T_x(X^\gamma) = T_x(X)^\gamma$ , as desired.  $\square$

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*E-mail address*, Adler: [jadler@american.edu](mailto:jadler@american.edu)

*E-mail address*, Lansky: [lansky@american.edu](mailto:lansky@american.edu)

(Adler,Lansky) DEPARTMENT OF MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY, WASHINGTON, DC 20016-8050